The Chain Rule

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Lesson Plan

- Why another differentiation rule?
- The formula for the chain rule
- Examples
Motivation

How to differentiate

- $\sqrt{1 + x^2}$
- $\sin(x^3)$
- $(1 + 2x)^{12}$
The Chain Rule

If \( g \) is differentiable at \( x \) and \( f \) is differentiable at \( y = g(x) \) then the composite function \( f \circ g \) is differentiable at \( x \) and

\[
(f \circ g)'(x) = f'(g(x)) \cdot g'(x)
\]

The verbal form of the chain rule

\( (f[g(x)])' = \text{outside}'[\text{inside}] \cdot \text{inside}' \)
Example 1 (A radical)

Differentiate $h(x) = \sqrt{x^2 + 1}$.

The "outside" function is $f(y) = \sqrt{y}$, the "inside" function is $g(x) = x^2 + 1$.

Then $f'(y) = \frac{1}{2\sqrt{y}}$; $g'(x) = 2x$ and

$$h'(x) = f'[g(x)] \cdot g'(x) = \frac{1}{2\sqrt{g(x)}} \cdot g'(x) = \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x = \frac{x}{\sqrt{x^2 + 1}}$$
Example 2 (A trigonometric function)

Differentiate $h(x) = \sin(x^3)$.

$$h'(x) = \left( \sin \left( x^3 \right) \right)' = \cos \left( x^3 \right) \cdot 3x^2$$

Therefore, $(\sin(x^3))' = 3x^2 \cos(x^3)$. 
Example 3 (A polynomial)

Differentiate \( h(x) = (1 + 2x)^{12} \).

\[
\begin{align*}
    h'(x) &= \left[ (1 + 2x)^{12} \right]' \\
    &= 12(1 + 2x)^{11} \cdot (1 + 2x)' \\
    &= 12(1 + 2x)^{11} \cdot 2 \\
    &= 24(1 + 2x)^{11}.
\end{align*}
\]
The derivative of $f \circ g$ can be written in Leibnitz notation. Recall that $h'(x) = \frac{dh}{dx}$. Denote $y = g(x)$, then

$$\frac{d}{dx}[f \circ g] = \frac{df}{dy} \cdot \frac{dy}{dx}$$
Example 4 (Nested chain rule)

Differentiate $h(x) = (\sqrt{x^4 + 2 + 1})^3$.

To differentiate $h(x)$ we proceed stepwise

$$h'(x) = \frac{d}{dx}(\sqrt{x^4 + 2 + 1})^3 = 3(\sqrt{x^4 + 2 + 1})^2 \cdot \frac{d}{dx}(\sqrt{x^4 + 2 + 1})$$

$$= 3(\sqrt{x^4 + 2 + 1})^2 \cdot \left(\frac{d}{dx}\sqrt{x^4 + 2 + 0}\right)$$

$$= 3(\sqrt{x^4 + 2 + 1})^2 \cdot \frac{1}{2\sqrt{x^4 + 2}} \cdot \frac{d}{dx}(x^4 + 2)$$

$$= 3(\sqrt{x^4 + 2 + 1})^2 \cdot \frac{1}{2\sqrt{x^4 + 2}} \cdot 4x^3$$

$$= \frac{6x^3(\sqrt{x^4 + 2 + 1})^2}{\sqrt{x^4 + 2}}$$
Example 5 (A function with parameters)

Find the derivative of \( h(x) = (ax^2 + 5)^n \) where \( a > 0 \) and \( n \) is a positive integer.

Here the outside function \( f(y) = y^n \) and the inside function \( g(x) = ax^2 + 5 \).

Since \( f'(y) = n y^{n-1} \) and \( g'(x) = 2ax \) we get

\[
h'(x) = n(ax^2 + 5)^{n-1} \cdot 2ax
\]
Example 6 (Differentiating a function that is not specified)

Suppose that \( f'(x) = 3x - 1 \). Find \( \frac{d}{dx}f(x^2) \) at \( x = 2 \).

The inside function is \( y = x^2 \), the outside function is \( f(y) \).

\[
\frac{d}{dx}f(x^2) = \frac{df}{dy} \cdot \frac{dy}{dx} = f'(y) \cdot 2x = (3y - 1) \cdot 2x = (3x^2 - 1) \cdot 2x
\]

Substituting \( x = 2 \) we get \( (3 \cdot 2^2 - 1)2 \cdot 2 = 44 \)
THE END