The Mean Value Theorem

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The Mean Value Theorem

Assume that $f$ is continuous on the closed interval $[a, b]$ and differentiable on $(a, b)$. Then there exists at least one value $c$ in $(a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

$f(b) - f(a)$ is the slope of the secant line.

$f'(c)$ is the slope of the tangent line.

**MVT:** There is at least one tangent line which is parallel to the secant line on $(a, b)$.

**Rolle's Theorem** is a special case of MVT where $f(a) = f(b)$. In this case $f'(c) = 0$. 

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Example 1

Find all the numbers $c$ that satisfy the conclusion of the Mean Value Theorem for $f(x) = x^3 - 2x$ on the interval $[-2, 2]$.

Since $f$ is a polynomial, it is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$.

By the MVT there is a number $c$ in $(-2, 2)$ such that $f'(c) = \frac{f(2) - f(-2)}{2 - (-2)}$.

Now $f(2) = 2^3 - 2 \cdot 2 = 4$, $f(-2) = (-2)^3 - 2 \cdot (-2) = -4$, and $f'(x) = 3x^2 - 2$.

So the equation becomes $3c^2 - 2 = \frac{4 - (-4)}{2 - (-2)} = \frac{8}{4} = 2$.

Therefore, $3c^2 - 2 = 2 \Rightarrow c^2 = \frac{4}{3} \Rightarrow c = \pm \frac{2}{\sqrt{3}}$.

Since both numbers $\frac{2}{\sqrt{3}}$ and $-\frac{2}{\sqrt{3}}$ are in $(-2, 2)$, we obtain $c = \pm \frac{2}{\sqrt{3}}$. 
Example 2

Suppose that \( f(0) = 7 \) and \( f'(x) \leq 4 \) for all values of \( x \). How large can \( f(6) \) possibly be?

We are given that \( f \) is differentiable (and therefore continuous) everywhere. We apply the Mean Value Theorem on \([0, 6]\). There is a number \( c \) in the interval \((0, 6)\) such that

\[
\frac{f(6) - f(0)}{6 - 0} = f'(c)
\]

We are given that \( f'(x) \leq 4 \) for all \( x \), so in particular \( f'(c) \leq 4 \).

Since \( f(0) = -7 \), the equation gives

\[
\frac{f(6) - (-7)}{6} \leq 4 \iff f(6) + 7 \leq 24 \implies f(6) \leq 17.
\]

The largest possible value of \( f(6) \) is 17.
Example 3

Does there exist a function \( f \) such that \( f(5) = 1 \), \( f(7) = 4 \), and \( f'(x) \leq 2 \) for all \( x \)?

Since the function \( f \) is differentiable everywhere, it has to satisfy the conclusion of the MVT on the interval \([5, 7]\). There must be a point \( c \) in \((5, 7)\) such that

\[
\frac{f(7) - f(5)}{7 - 5} = f'(c)
\]

We are given that \( f'(x) \leq 2 \) for all \( x \), so in particular \( f'(c) \leq 2 \).

Since \( f(5) = 1 \), \( f(7) = 4 \), the equation gives

\[
\frac{4 - (-1)}{7 - 5} \leq 2 \quad \Rightarrow \quad \frac{5}{2} \leq 2
\]

Contradiction!

Therefore, the answer is \( \boxed{No} \), there is no function such that \( f(5) = 1 \), \( f(7) = 4 \), and \( f'(x) \leq 2 \) for all \( x \).
If $f'(x) = 0$ for all $x$ in an interval $(a, b)$ then $f$ is constant on $(a, b)$.

Let $x_1$ and $x_2$ be any two numbers in $(a, b)$ with $x_1 < x_2$. By applying the MVT to $f$ on the interval $[x_1, x_2]$ we get a number $c$ in $(x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

Since $f'(x) = 0$ for all $x$ in $(a, b)$, we have $f'(c) = 0$ and equation becomes

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0 \quad \Rightarrow \quad f(x_2) - f(x_1) = 0 \quad \Rightarrow \quad f(x_2) = f(x_1)$$

Therefore $f$ has the same value at any two numbers in $(a, b)$. This means that $f$ is constant on $(a, b)$. 

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THE END