Related Rates

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In related rates problems we compute the rate of change of one quantity in terms of the rate of change of the other.

**Rate of change = derivative**

**Strategy:**
- Draw a diagram and introduce the notation
- Write an equation relating the quantities of the problem
- Take derivative with respect to time (Use the Chain Rule!)
- Substitute the given information and solve for the unknown.
Example 1

The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

Given: \( \frac{dx}{dt} = 8, \quad \frac{dy}{dt} = 3 \)

Find: \( \frac{dA}{dt} \) when \( x = 20 \) and \( y = 10 \).

Area of the rectangle: \( A = x \cdot y \)

Differentiate with respect to \( t \) : \( \frac{dA}{dt} = \frac{dx}{dt} y + x \frac{dy}{dt} \)

Substitute the given information: \( \frac{dA}{dt} = 8 \cdot 10 + 20 \cdot 3 = 120 \text{ cm}^2/\text{s} \)
Example 2

At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 PM?

Notation: 
- $t$ - time elapsed since noon,
- $x$ - distance travelled by A since noon,
- $y$ - distance travelled by B since noon,
- $z$ - distance between A and B

Given: $\frac{dx}{dt} = 35$, $\frac{dy}{dt} = 25$.  
Find: $\frac{dz}{dt}$ when $t = 4$.

The Pythagorean theorem:  $z^2 = (150 - x)^2 + y^2$
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Given: \( \frac{dx}{dt} = 35, \quad \frac{dy}{dt} = 25. \) Find: \( \frac{dz}{dt} \) when \( t = 4. \)

The Pythagorean theorem: \( z^2 = (150 - x)^2 + y^2 \)

Differentiate: \( 2z \frac{dz}{dt} = 2(150 - x) \left( -\frac{dx}{dt} \right) + 2y \frac{dy}{dt} \)

At time \( t = 4: \) \( x = 35 \cdot 4 = 140 \) \( y = 25 \cdot 4 = 100 \)

\( z^2 = (150 - 140)^2 + 100^2 = 10100, \) so \( z = 10\sqrt{101}. \)

Plug in: \( 2 \cdot 10\sqrt{101} \frac{dz}{dt} = 2(150 - 140)(-35) + 2 \cdot 100 \cdot 25 \)

Solve for \( \frac{dz}{dt}: \) \( \frac{dz}{dt}(4) = \frac{430}{\sqrt{101}} \text{km/h} \)
Example 3

A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string has been let out?

Notation: $t$ - time elapsed,  
$x$ - distance travelled by the kite,  
$y$ - length of the string,  
$\theta$ - angle between the string and the horizontal

Given: $\frac{dx}{dt} = 8$. Find $\frac{d\theta}{dt}$ when $y = 200$.

$\tan \theta = \frac{100}{x}$  
Differentiate: $\sec^2 \theta \cdot \frac{d\theta}{dt} = -\frac{100}{x^2} \cdot \frac{dx}{dt}$
Example 3

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Given: \( \frac{dx}{dt} = 8 \), Find: \( \frac{d\theta}{dt} \) when \( y = 200 \).

\[
\tan \theta = \frac{100}{x}
\]
Differentiate:
\[
\sec^2 \theta \cdot \frac{d\theta}{dt} = -\frac{100}{x^2} \cdot \frac{dx}{dt}
\]

When \( y = 200 \):
\[
x^2 = 200^2 - 100^2 = 30000
\]
\[
\sec \theta = \frac{y}{x} \implies \sec^2 \theta = \frac{y^2}{x^2} = \frac{40000}{30000} = \frac{4}{3}
\]
Plug in:
\[
\frac{4}{3} \cdot \frac{d\theta}{dt} = -\frac{100}{30000} \cdot 8 \implies \frac{d\theta}{dt} = -0.02 \text{ rad/s}
\]
Example 4

The top of a ladder slides down a vertical wall at a rate of 0.15 m/s. At the moment when the bottom of the ladder is 3 m from the wall, it slides away from the wall at a rate of 0.2 m/s. How long is the ladder?

Notation: $t$ - time elapsed,
$x$ - distance of the bottom to the wall,
$y$ - distance of the top to the ground,
l - length of the ladder

Given: $\frac{dy}{dt} = -0.15$ and $\frac{dx}{dt} = 0.2$ when $x = 3$. Find: l.

$l^2 = x^2 + y^2$  
Differentiate: $0 = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$

Plug in: $0 = 2 \cdot 3 \cdot 0.2 + 2y(-0.15) \implies y = 4$

$l^2 = 3^2 + 4^2 = 25 \implies l = 5 \text{ m}$
Example 5

Water pours into a fish tank at a rate of 3 ft$^3$/min. How fast is the water level rising if the base of the tank is a rectangle of dimensions 2 \times 3 \text{ ft}?

Notation: $t$ - time elapsed, $h$ - height of the water, $V$ - volume of the water

Given: $\frac{dV}{dt} = 3$. Find: $\frac{dh}{dt}$.

$V = 3 \cdot 2 \cdot h = 6h$  Differentiate: $\frac{dV}{dt} = 6 \frac{dh}{dt}$

Plug in: $3 = 6 \cdot \frac{dh}{dt} \implies \frac{dh}{dt} = 0.5 \text{ ft/min}$
THE END