Antiderivatives

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Definition

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A function F is called an antiderivative of a function f on an interval I if $F'(x) = f(x)$ for all x in I.

$$f(x) = 2x \implies F(x) = x^2 \quad \text{because} \quad F'(x) = (x^2)' = 2x$$
$$\implies F(x) = x^2 + 5 \quad \text{because} \quad F'(x) = (x^2+5)' = 2x$$
$$\implies F(x) = x^2 + C \quad \text{because} \quad F'(x) = (x^2+C)' = 2x$$

If $F(x)$ is an antiderivative of f then $F(x) + C$ is also an antiderivative of f.
Moreover, any other antiderivative of f has this form.
$F(x) + C$ is called the general antiderivative of f.

Antiderivatives

Notation

$\int f(x)\,dx = F(x) + C$ means that $F'(x) = f(x)$. We say that $\int f(x)\,dx$ is the general antiderivative or indefinite integral of f.

1. $\int 2x\,dx = x^2 + C$
2. $\int \sin x\,dx = -\cos x + C$ because $(-\cos x + C)' = \sin x$
3. $\int x^3\,dx = \frac{x^4}{4} + C$ because $\left(\frac{x^4}{4} + C\right)' = x^3$
4. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ because $\left(\frac{x^{n+1}}{n+1} + C\right)' = \frac{(n+1)x^n}{n+1} = x^n$

A Table of General Antiderivatives

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1. $$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$$
2. $$\int k\,dx = kx + C,$$ where k is a constant
3. $$\int \sin x\,dx = -\cos x + C$$
4. $$\int \cos x\,dx = \sin x + C$$
5. $$\int \sec^2 x\,dx = \tan x + C$$
6. $$\int \csc^2 x\,dx = -\cot x + C$$

Rules of Integration

Rules of Integration

1. $$\int kf(x)\,dx = k\int f(x)\,dx$$
2. $$\int f(x)\pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

- $$\int 5\sqrt{x}\,dx = 5\int \sqrt{x}\,dx = 5\int x^{\frac{1}{2}}\,dx = 5\cdot\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{10}{3}x^{\frac{3}{2}} + C$$
- $$\int (\cos x + \frac{1}{x^2})\,dx = \int \cos x\,dx + \int x^{-2}\,dx = \sin x + \frac{x^{-2+1}}{-2+1} + C = \sin x + \frac{1}{x} + C$$
- $$\int (7x^4 + 5x - 2)\,dx = 7\int x^4\,dx + 5\int x\,dx - \int 2\,dx = 7\frac{x^5}{5} + 5\frac{x^2}{2} - 2x + C$$

Example 1

Evaluate $\int x^3(x^2+2)^2\,dx$

$$\begin{aligned}\int x^3(x^2+2)^2\,dx &= \int x^3(x^4+2x^2+4)\,dx \\ &= \int (x^7+2x^5+4x^3)\,dx \\ &= \frac{x^8}{8}+2\frac{x^6}{6}+4\frac{x^4}{4}+C \\ &= \boxed{\frac{x^8}{8}+\frac{x^6}{3}+x^4+C}\end{aligned}$$

Example 2

Evaluate $\int \frac{1+2x^5-4x^7}{x^4}\,dx$

$$\begin{aligned}
\int \frac{1+2x^5-4x^7}{x^4}\,dx &= \int \left(\frac{1}{x^4}+\frac{2x^5}{x^4}-\frac{4x^7}{x^4}\right)dx \\
&= \int (x^{-4}+2x-4x^3)\,dx \\
&= \frac{x^{-3}}{-3}+2\frac{x^2}{2}-4\frac{x^4}{4}+C \\
&= \boxed{-\frac{1}{3x^3}+x^2-x^4+C}
\end{aligned}$$

Example 3

Evaluate $\displaystyle\int \frac{\sec x + \cos x}{\cos x}\, dx$

$$\begin{aligned} \int \frac{\sec x + \cos x}{\cos x}\, dx &= \int \left(\frac{\sec x}{\cos x} + \frac{\cos x}{\cos x} \right) dx \\ &= \int (\sec^2 x + 1)\, dx \\ &= \boxed{\tan x + x + C} \end{aligned}$$

Example 4

Evaluate $\int \tan^2 x \, dx$

$$\int \tan^2 x \, dx = \int \left(\sec^2 x - 1\right) \, dx$$
$$= \boxed{\tan x - x + C}$$

Example 5

Find f if $f'' = \cos x$

$$f'(x) = \int \cos x \, dx = \sin x + C$$

$$f(x) = \int (\sin x + C) = -\cos x + Cx + D$$

$$\boxed{f(x) = -\cos x + Cx + D}$$

Example 6

Find f if $f'' = \frac{3}{\sqrt{x}}$ and $f(4) = 20$, $f'(4) = 7$.

$$f'(x) = \int \frac{3}{\sqrt{x}}\,dx = 3\int x^{-\frac{1}{2}}\,dx = 3\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = 6x^{\frac{1}{2}} + C$$

$f'(4) = 7 \quad \Rightarrow \quad 6(4)^{\frac{1}{2}} + C = 7 \quad \Rightarrow \quad C = 7 - 12 = -5.$ Therefore,
$f'(x) = 6x^{\frac{1}{2}} - 5$

$$f(x) = \int (6x^{\frac{1}{2}} - 5)\,dx = 6\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 5x + D = 4x^{\frac{3}{2}} - 5x + D$$

$f(4) = 20 \quad \Rightarrow \quad 4(4)^{\frac{3}{2}} - 5 \cdot 4 + D = 20 \quad \Rightarrow \quad D = 20 - 12 = 8.$

Therefore, $\boxed{f(x) = 4x^{\frac{3}{2}} - 5x + 8}$

THE END