A Catalog of Essential Functions

Tamara Kucherenko

Lesson Plan

- Linear functions GO
- Polynomials GO
- Power functions GO
- Rational functions GO
- Trigonometric functions GO

Linear Functions

A linear function is a function of the form

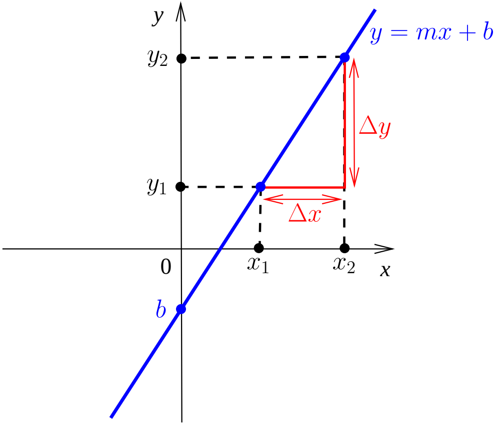
$$f(x) = mx + b \quad (m, b \text{ are constants})$$

The graph of f is a line with slope m and y-intercept b.

Change in x is $\Delta x = x_2 - x_1$
Change in y is $\Delta y = f(x_2) - f(x_1)$

The slope m of a line is

$$m = \frac{\Delta y}{\Delta x} = \frac{rise}{run}$$



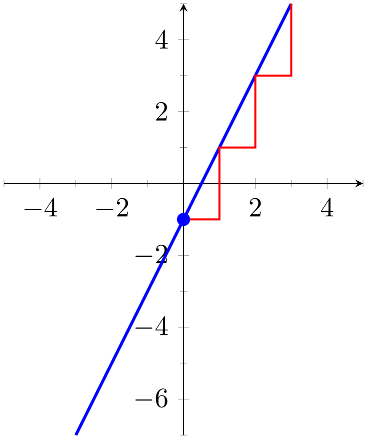
Example

Graph $y = 2x - 1$

There are several ways to graph a line

- Using the slope and y-intercept

  Start with y-intercept (0,-1)
  When x increases by 1, y increases by 2

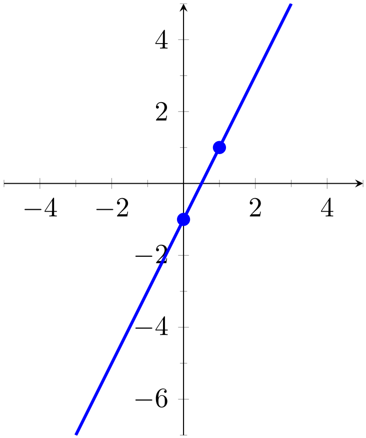


Example

Graph $y = 2x - 1$

There are several ways to graph a line

- Using the slope and y-intercept
- Using any two points
  $x = 0, \quad y = -1$
  $x = 1, \quad y = 1$

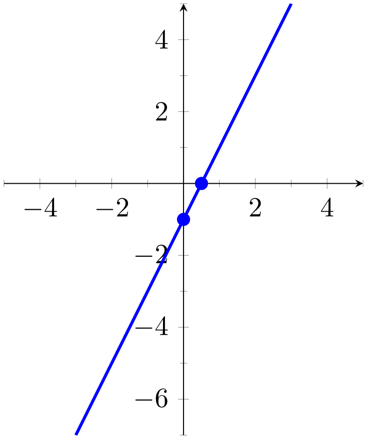


Example

Graph $y = 2x - 1$

There are several ways to graph a line

- Using the slope and y-intercept
- Using any two points
- Using y-intercept and x-intercept
  $x = 0, \quad y = -1$
  $y = 0, \quad x = 1/2$



Useful Facts

- The larger the absolute value of the slope $|m|$, the steeper the line.
- If $m > 0$ the line $y = mx + b$ slants upward from left to right.
- If $m < 0$ the line $y = mx + b$ slants downward from left to right.
- The horizontal line $y = b$ has slope $m = 0$
- The vertical line $x = c$ is not a function and its slope is undefined.
- Lines with slopes m_1 and m_2 are parallel if and only if $m_1 = m_2$
- Lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 = -1/m_2$
- The line through (a, b) with slope m has equation

$$\boxed{y - b = m(x - a)} \quad \text{(point-slope form)}$$

Polynomials

A polynomial is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

- n is a nonnegative integer called the degree of $f(x)$ (assuming that $a_n \neq 0$).
- The numbers $a_n, a_{n-1}, ..., a_1, a_0$ are called coefficients.
- a_n is called the leading coefficient.
- The domain of $f(x)$ is $\mathbb{R}$

Polynomials

- If $\underline{n=0}$, $f(x) = a_0$ is a constant function.
- If $\underline{n=1}$, $f(x) = a_1 x + a_0$ is a linear function.
- If $\underline{n=2}$, $f(x) = a_2 x^2 + a_1 x + a_0$ is a quadratic function.
- If $\underline{n=3}$, $f(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$ is a cubic function.

Quadratic Functions

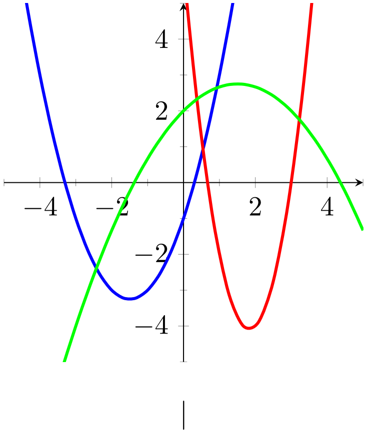
A quadratic function is defined by

$$f(x) = ax^2 + bx + c$$

a, b, c are constants with $a \neq 0$

The graph of f is a parabola.

$$y = x^2 + 3x - 1$$
$$y = 3x^2 - 11x + 6$$
$$y = -\frac{1}{3}x^2 + x + 2$$



Quadratic Functions

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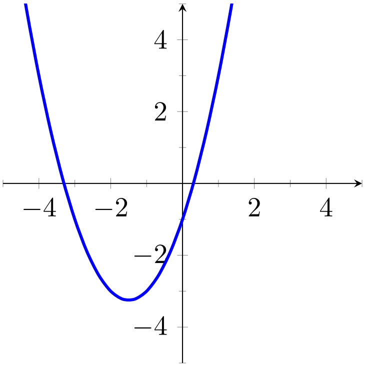
The graph of f is a parabola.

- If $a > 0$, then the parabola opens upward



$y = x^2 + 3x - 1$

Quadratic Functions

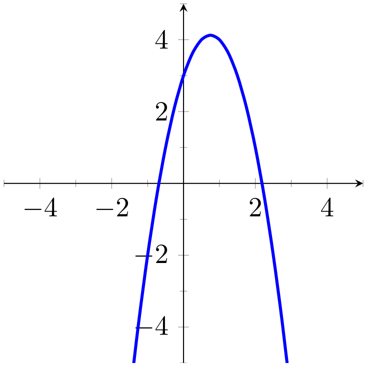
A quadratic function is defined by

$$f(x) = ax^2 + bx + c$$

a, b, c are constants with $a \neq 0$

The graph of f is a parabola.

- If $a > 0$, then the parabola opens upward
- If $a < 0$, then the parabola opens downward



$$y = -2x^2 + 3x + 3$$

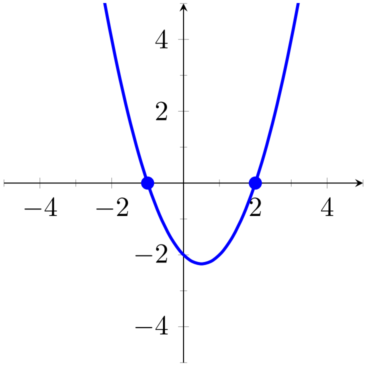
Quadratic Functions

The roots of $f(x) = ax^2 + bx + c$ are given by the quadratic formula

$$\text{Roots of } f(x) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quantity $D = b^2 - 4ac$ is called the discriminant of $f(x)$.

- If $D > 0$, $f(x)$ has two real roots.



$y = x^2 - x - 2$

$D = (-1)^2 - 4 \cdot 1 \cdot (-2) = 9 > 0$

Quadratic Functions

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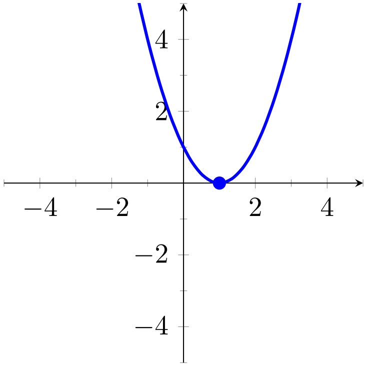
The quantity $D = b^2 - 4ac$ is called the discriminant of $f(x)$.

- If $D > 0$, $f(x)$ has two real roots.
- If $D = 0$, $f(x)$ has one real root (a "double root").



$$y = x^2 - 2x + 1$$
$$D = (-2)^2 - 4 \cdot 1 \cdot 1 = 0$$

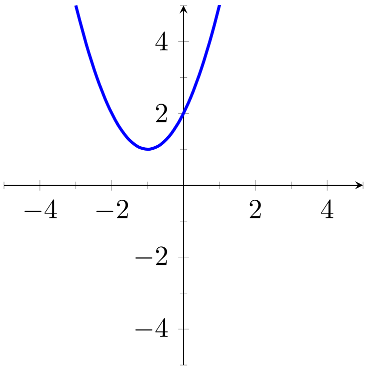
Quadratic Functions

The roots of $f(x) = ax^2 + bx + c$ are given by the quadratic formula

$$\text{Roots of } f(x) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quantity $D = b^2 - 4ac$ is called the discriminant of $f(x)$.

- If $D > 0$, $f(x)$ has two real roots.
- If $D = 0$, $f(x)$ has one real root (a "double root").
- If $D < 0$, $f(x)$ has no real roots.



$$y = x^2 + 2x + 2$$
$$D = 2^2 - 4 \cdot 1 \cdot 2 = -4 < 0$$

Quadratic Functions

When $f(x) = ax^2 + bx + c$ has two real roots r_1 and r_2, then $f(x)$ factors as

$$\boxed{f(x) = a(x - r_1)(x - r_2)}$$

Example: $f(x) = 2x^2 - 3x + 1$ has roots $r_{1,2} = \frac{3 \pm \sqrt{9-8}}{4} = \frac{3 \pm 1}{4}$ or $r_1 = 1$ and $r_2 = \frac{1}{2}$. Therefore,

$$f(x) = 2(x - 1)(x - \frac{1}{2})$$

There is a simple relation between the roots of a quadratic polynomial and its coefficients (Vieta's formulas)

$$\boxed{r_1 + r_2 = -\frac{b}{a} \quad \text{and} \quad r_1 \cdot r_2 = \frac{c}{a}}$$

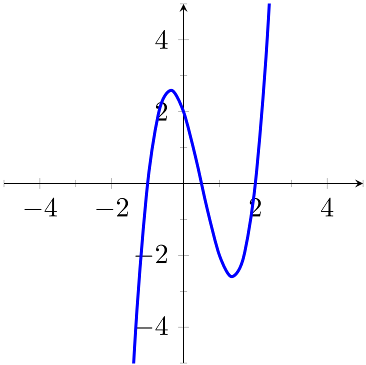
Cubic Functions

A cubic function is defined by

$$f(x) = ax^3 + bx^2 + cx + d$$

a, b, c, d are constants with $a \neq 0$

- If $a > 0$, then the graph of f goes from the third quadrant to the first



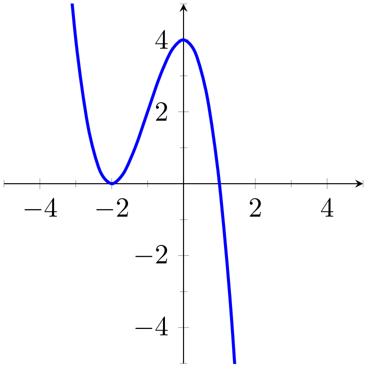
$$y = 2x^3 - 3x^2 - 3x + 2$$

Cubic Functions

A cubic function is defined by

$$f(x) = ax^3 + bx^2 + cx + d$$
a, b, c, d are constants with $a \neq 0$

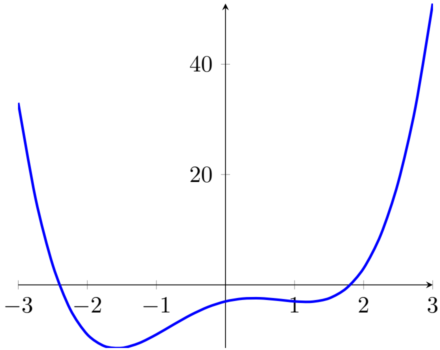
- If $a > 0$, then the graph of f goes from the third quadrant to the first
- If $a < 0$, then the graph of f goes from the second quadrant to the fourth



$y = -x^3 - 3x^2 + 4$

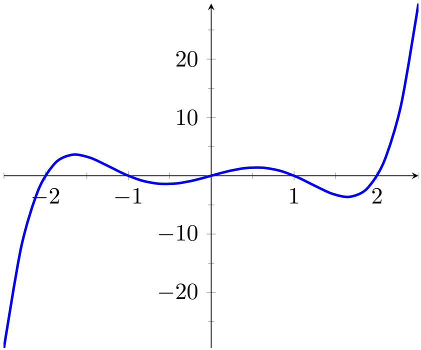
Higher Degree Polynomials

A polynomial of degree 4



$$y = x^4 - 4x^2 + 3x - 3$$

A polynomial of degree 5



$$y = x^5 - 5x^3 + 4x$$

Power Functions

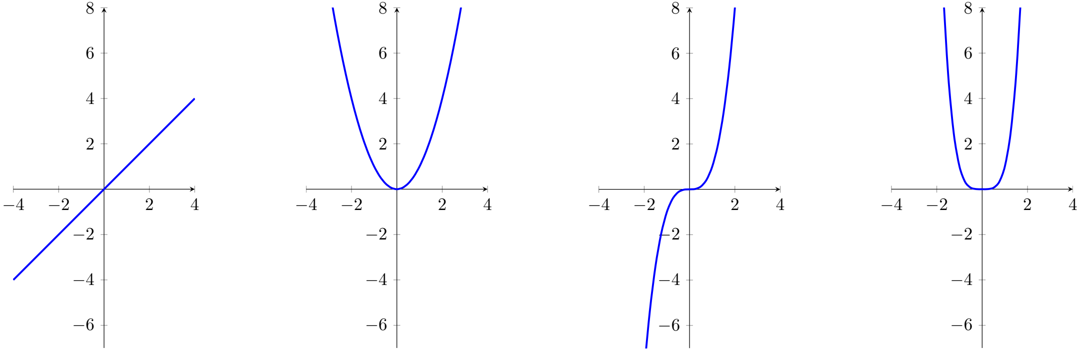
A power function is a function of the form

$$f(x) = x^a, \quad (a \text{ is a constant})$$

We will consider three special cases: $a = n, a = \frac{1}{n}$, and $a = -n$, where n is a positive integer.

$f(x) = x^n$

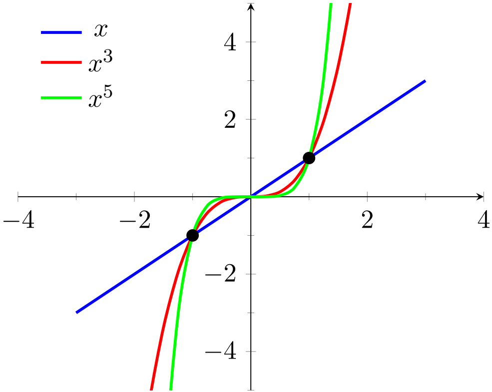
$n = 1$ $\quad$ $n = 2$ $\quad$ $n = 3$ $\quad$ $n = 4$



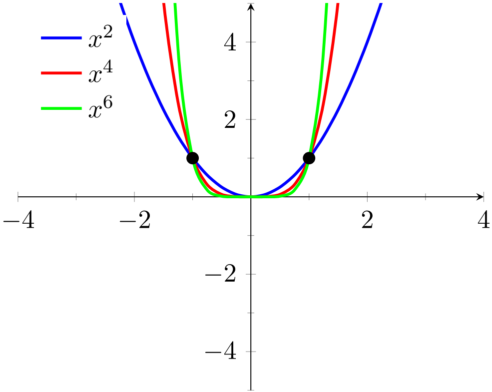
$f(x) = x$ $\quad$ $f(x) = x^2$ $\quad$ $f(x) = x^3$ $\quad$ $f(x) = x^4$

$f(x) = x^n$

n is odd: $(-x)^n = -x^n$, so $f(x)$ is odd

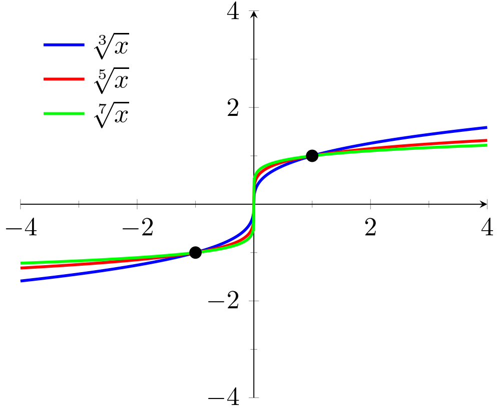


n is even: $(-x)^n = x^n$, so $f(x)$ is even

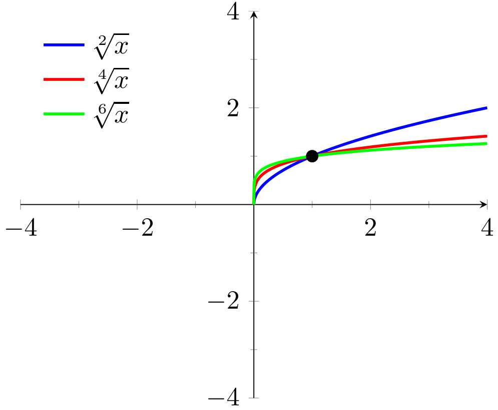


$f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$

n is odd: domain of $\sqrt[n]{x}$ is $\mathbb{R}$



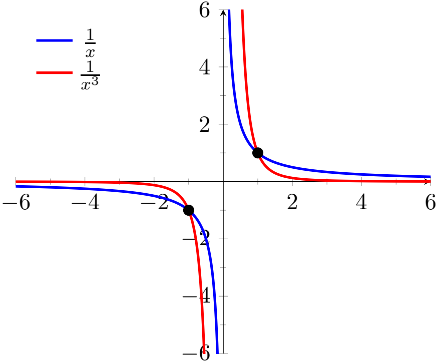
n is even: domain of $\sqrt[n]{x}$ is $[0, \infty)$



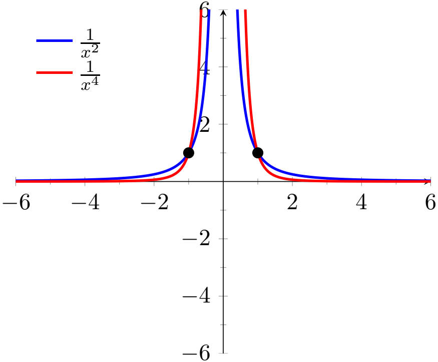
$f(x) = x^{-n} = \frac{1}{x^n}$

Domain of $\frac{1}{x^n}$ is $(-\infty, 0) \cup (0, \infty)$. The graph is a hyperbola.

n is odd: $f(x) = \frac{1}{x^n}$ is odd



n is even: $f(x) = \frac{1}{x^n}$ is even



Rational Functions

A rational function is quotient of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}, \quad (P(x), Q(x) \text{ are polynomials})$$

The domain of a rational function $\frac{P(x)}{Q(x)}$ is the set of numbers x such that $Q(x) \neq 0$.

Example:

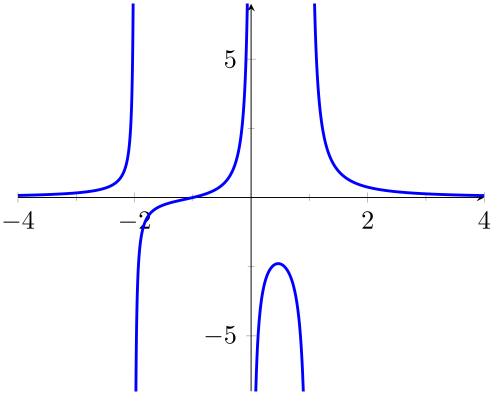
$$f(x) = \frac{x+1}{x^3 + x^2 - 2x}$$

The denominator cannot be zero, so we set

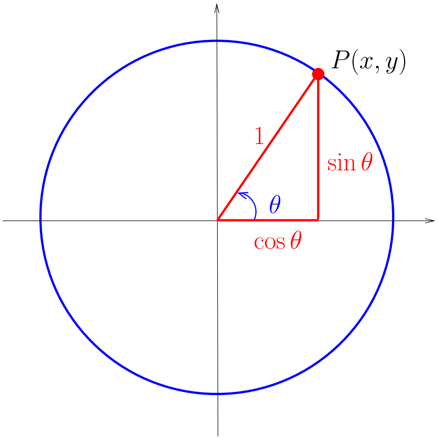
$$x^3 + x^2 - 2x \neq 0$$
$$x(x^2 + x - 2) \neq 0$$
$$x(x+2)(x-1) \neq 0$$
$$x \neq 0,\ x \neq -2,\ \text{and}\ \ x \neq 1$$

Domain is

$$\boxed{(-\infty, -2) \cup (-2, 0) \cup (0, 1) \cup (1, \infty)}$$



Trigonometric Functions

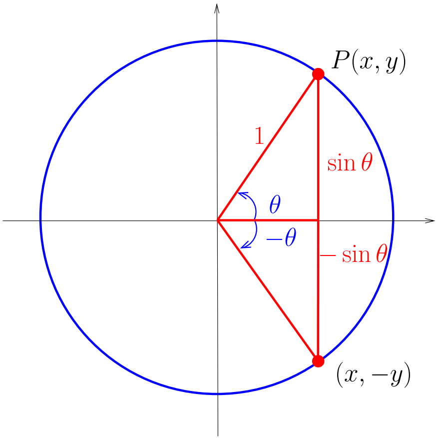


Let $P(x, y)$ be the point on the unit circle corresponding to the angle θ.
We define

$$\cos\theta = x\text{-coordinate of } P$$
$$\sin\theta = y\text{-coordinate of } P$$

Note that we always measure angles in radians.

Trigonometric Functions



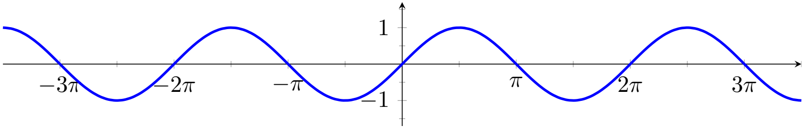
Let $P(x, y)$ be the point on the unit circle corresponding to the angle θ.
We define

$$\cos\theta = x\text{-coordinate of } P$$
$$\sin\theta = y\text{-coordinate of } P$$

Note that we always measure angles in radians.
Consider angle $-\theta$.
$\cos(-\theta) = \cos(\theta)$, $\sin(-\theta) = -\sin(\theta)$.
Thus, $\cos(\theta)$ is an even function and $\sin(\theta)$ is an odd function.
Also, $\cos\theta$ and $\sin\theta$ are periodic with period 2π.

Trigonometric Functions

Graph of $y = \sin x$



Graph of $y = \cos x$

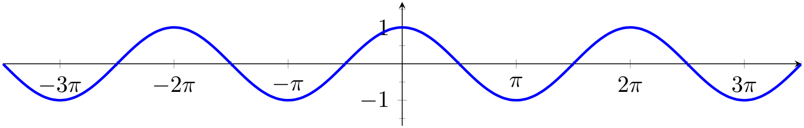


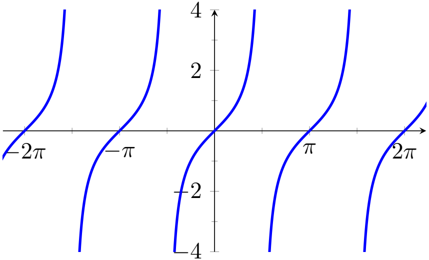
Table of Values

θ	$\sin\theta$	$\cos\theta$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0

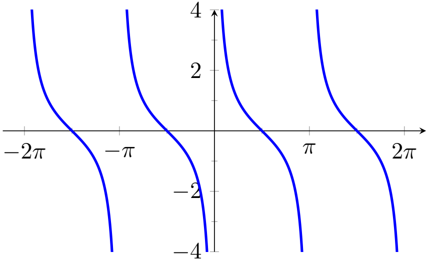
Trigonometric Functions

There are four other trigonometric functions, each defined in terms of $\sin x$ and $\cos x$.

Tangent: $\tan x = \frac{\sin x}{\cos x}$



Cotangent: $\cot x = \frac{\cos x}{\sin x}$

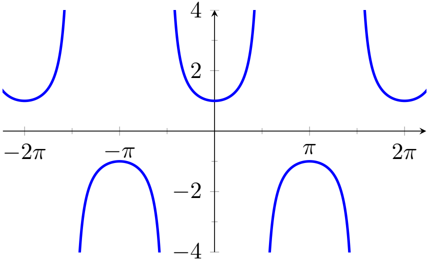


Tangent and cotangent are periodic with period π.

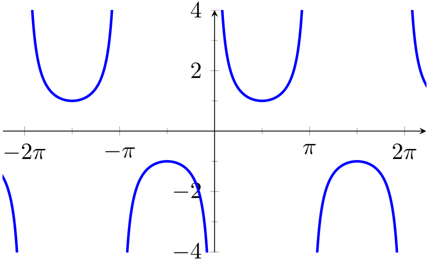
Trigonometric Functions

There are four other trigonometric functions, each defined in terms of $\sin x$ and $\cos x$.

Secant: $\sec x = \frac{1}{\cos x}$



Cosecant: $\csc x = \frac{1}{\sin x}$



Secant and cosecant are periodic with period 2π.

Trigonometric Functions

Trigonometric Identities

- *Pythagorean Theorem:* $\sin^2 x + \cos^2 x = 1$
- *Equivalent versions:* $\tan^2 x + 1 = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x$
- *Double-angle formulas:*
  $\sin 2x = 2 \sin x \cos x, \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x), \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- *Addition formulas:*
  $\sin(x + y) = \sin x \cos y + \cos x \sin y, \quad \cos(x + y) = \cos x \cos y - \sin x \sin y$
- *Shift formulas:* $\sin(x + \frac{\pi}{2}) = \cos x, \quad \cos(x + \frac{\pi}{2}) = -\sin x$

THE END