The Chain Rule

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Lesson Plan

- Why another differentiation rule?
- The formula for the chain rule
- Examples

Motivation

How to differentiate

- $\sqrt{1+x^2}$
- $\sin(x^3)$
- $(1+2x)^{12}$

The Chain Rule

The Chain Rule.

If g is differentiable at x and f is differentiable at $y = g(x)$ then the composite function $f \circ g$ is differentiable at x and $\boxed{(f \circ g)'(x) = f'(g(x)) \cdot g'(x)}$

(	f	$[g(x)]$	$)'$	$=$	f'	$[g(x)]$	$\cdot$	$g'(x)$
	↑	↑			↑	↑		↑
	outside	inside			derivative of the outside	inside left along		derivative of the inside

The verbal form of the chain rule $(f[g(x)])' = \text{outside}'[\text{inside}] \cdot \text{inside}'$

Example 1 (*A radical*)

Differentiate $h(x) = \sqrt{x^2+1}$.

The "outside" function is $f(y) = \sqrt{y}$, the "inside" function is $g(x) = x^2 + 1$.

$$\text{Then} \quad f'(y) = \frac{1}{2\sqrt{y}}; \quad g'(x) = 2x \quad \text{and}$$

$$h'(x) = f'[g(x)] \cdot g'(x) = \frac{1}{2\sqrt{g(x)}} \cdot g'(x) = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \boxed{\frac{x}{\sqrt{x^2+1}}}$$

Example 2 (*A trigonometric function*)

Differentiate $h(x) = \sin(x^3)$.

$$h'(x) = \Big(\underset{\substack{\uparrow \\ \text{outside}}}{\sin} \quad \underset{\substack{\uparrow \\ \text{inside}}}{(x^3)} \Big)' = \underset{\substack{\uparrow \\ \text{derivative} \\ \text{of the outside}}}{\cos} \quad \underset{\substack{\uparrow \\ \text{inside} \\ \text{left along}}}{(x^3)} \cdot \underset{\substack{\uparrow \\ \text{derivative} \\ \text{of the inside}}}{3x^2}$$

Therefore, $(\sin(x^3))' = \boxed{3x^2\cos(x^3)}$.

Example 3 (*A polynomial*)

Differentiate $h(x) = (1+2x)^{12}$.

$$h'(x) = [\ \underbrace{\overset{\overset{\text{inside}}{\downarrow}}{(1+2x)}^{12}}_{\overset{\uparrow}{\text{outside}}}\ ]' = \underbrace{12\overset{\overset{\text{inside}}{\downarrow}}{(1+2x)}^{11}}_{\overset{\uparrow}{\text{derivative of the outside}}} \cdot \underbrace{2}_{\overset{\uparrow}{\text{derivative of the inside}}}$$

Therefore, $((1+2x)^{12})' = \boxed{24(1+2x)^{11}}$.

Leibnitz Notation

The derivative of $f \circ g$ can be written in Leibnitz notation.

Recall that $h'(x) = \frac{dh}{dx}$. Denote $y = g(x)$, then

$$\frac{d}{dx}[f \circ g] = \frac{df}{dy} \cdot \frac{dy}{dx}$$

Example 4 (*Nested chain rule*)

Differentiate $h(x) = (\sqrt{x^4+2}+1)^3$.

To differentiate $h(x)$ we proceed stepwise

$$\begin{aligned} h'(x) = \frac{d}{dx}(\sqrt{x^4+2}+1)^3 &= 3(\sqrt{x^4+2}+1)^2 \cdot \frac{d}{dx}(\sqrt{x^4+2}+1) \\ &= 3(\sqrt{x^4+2}+1)^2 \cdot (\frac{d}{dx}\sqrt{x^4+2}+0) \\ &= 3(\sqrt{x^4+2}+1)^2 \cdot \frac{1}{2\sqrt{x^4+2}} \cdot \frac{d}{dx}(x^4+2) \\ &= 3(\sqrt{x^4+2}+1)^2 \cdot \frac{1}{2\sqrt{x^4+2}} \cdot 4x^3 \\ &= \boxed{\frac{6x^3(\sqrt{x^4+2}+1)^2}{\sqrt{x^4+2}}} \end{aligned}$$

Example 5 (*A function with parameters*)

Find the derivative of $h(x) = (ax^2 + 5)^n$ where $a > 0$ and n is a positive integer.

Here the outside function $f(y) = y^n$ and the inside function $g(x) = ax^2 + 5$.
Since $f'(y) = ny^{n-1}$ and $g'(x) = 2ax$ we get

$$\boxed{h'(x) = n(ax^2 + 5)^{n-1} \cdot 2ax}$$

Example 6 (*Differentiating a function that is not specified*)

Suppose that $f'(x) = 3x - 1$. **Find** $\frac{d}{dx}f(x^2)$ **at** $x = 2$.

The inside function is $y = x^2$, the outside function is $f(y)$.

$$\frac{d}{dx}f(x^2) = \frac{df}{dy} \cdot \frac{dy}{dx} = f'(y) \cdot 2x = (3y - 1) \cdot 2x = (3x^2 - 1) \cdot 2x$$

Substituting $x = 2$ we get $(3 \cdot 2^2 - 1)2 \cdot 2 = \boxed{44}$

THE END