Continuity

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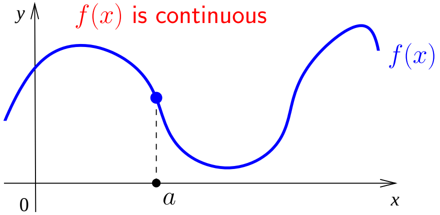
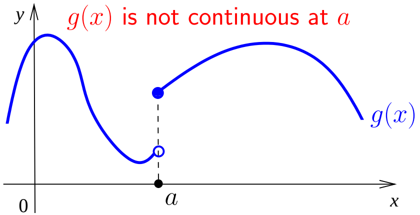
Lesson Plan

- Continuity at a Point GO
- Examples of Discontinuities GO
- Continuity of Basic Functions GO
- Intermediate Value Theorem GO

Continuity at a Point





Definition

A function f is continuous at a number a if

1. $f(a)$ *is defined*
2. $\lim_{x \to a} f(x)$ *exists*
3. $\lim_{x \to a} f(x) = f(a)$

Example 1

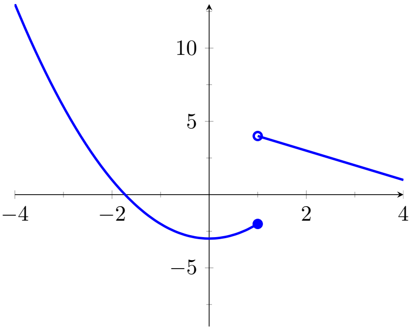
Is $f(x) = \begin{cases} x^2 - 3 & \text{if } x \le 1 \\ -x + 5 & \text{if } x > 1 \end{cases}$ continuous at $x = 1$?

Solution:

1. f is defined at $x = 1$, $f(1) = -2$
2. $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 - 3) = -2$
   $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (-x + 5) = 4$

Thus, $\lim_{x \to 1} f(x)$ does not exist and

$\boxed{f \text{ is not continuous at } x = 1}$



We say that f is discontinuous at a (or has a discontinuity at a) if f is not continuous at a.

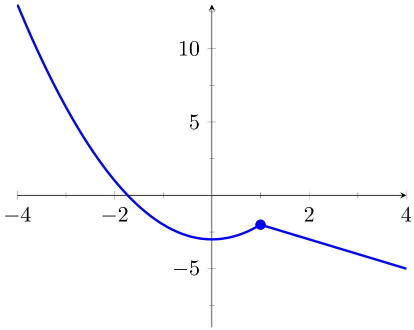
Example 2

Is $f(x) = \begin{cases} x^2 - 3 & \text{if } x \leq 1 \\ -x - 1 & \text{if } x > 1 \end{cases}$ continuous at $x = 1$?

Solution:

1. f is defined at $x = 1$, $f(1) = -2$
2. $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 - 3) = -2$
   $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (-x - 1) = -2$
   Thus, $\lim_{x \to 1} f(x) = -2$
3. $\lim_{x \to 1} f(x) = -2 = f(1)$

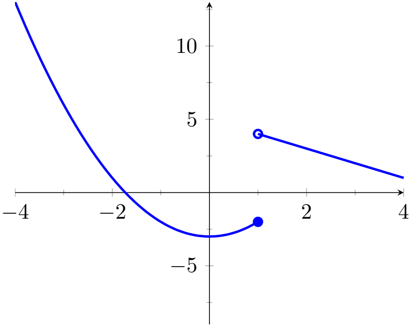
Therefore, $\boxed{f \text{ is continuous at } x = 1}$

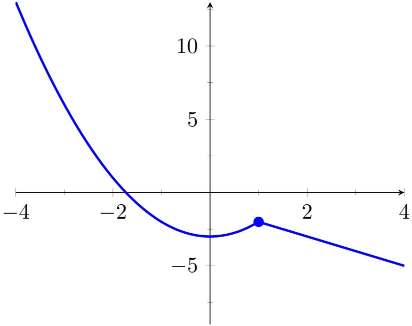


Example 1 vs. Example 2

Example 1: $f(x) = \begin{cases} x^2 - 3 & \text{if } x \leq 1 \\ -x + 5 & \text{if } x > 1 \end{cases}$

Example 2: $f(x) = \begin{cases} x^2 - 3 & \text{if } x \leq 1 \\ -x - 1 & \text{if } x > 1 \end{cases}$





Example 3

For what value of the constant c is

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$ continuous at $x = 2$?

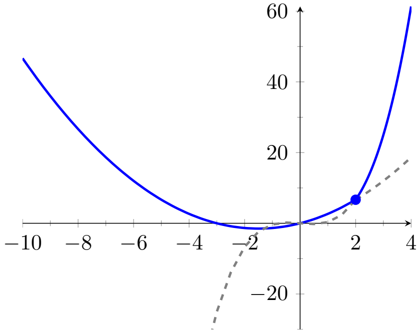
Solution:

1. f is defined at $x = 2$, $f(2) = 8 - 2c$
2. $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} (cx^2 + 2x) = 4c + 4$

   $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (x^3 - cx) = 8 - 2c$

   $\lim\limits_{x\to 2} f(x)$ exist if $4c + 4 = 8 - 2c$ or $c = \frac{2}{3}$.
3. $\lim\limits_{x\to 2} f(x) = 4 \cdot \frac{2}{3} + 4 = \frac{20}{3} = f(2)$

Therefore, f is continuous at 2 when $\boxed{c = \frac{2}{3}}$



$$f(x) = \begin{cases} \frac{2}{3}x^2 + 2x & \text{if } x < 2 \\ x^3 - \frac{2}{3}x & \text{if } x \geq 2 \end{cases}$$

Examples of Discontinuities

Recall that $f(x)$ is continuous if

1. $f(a)$ is defined **2.** $\lim_{x \to a} f(x)$ exists **3.** They are equal.

If one of them is violated, f is has a discontinuity at a.

There are three common types of discontinuities:

- Removable Discontinuity ($\lim_{x \to a} f(x)$ exists but does not equal $f(a)$)
- Jump Discontinuity (one-sided limits both exist but are not equal)
- Infinite discontinuity (at least one of the one-sided limits is infinite)

Remark: $f(x) = \sin \frac{1}{x}$ has a discontinuity at $x = 0$ which is none of these three types.

Removable Discontinuity

Consider

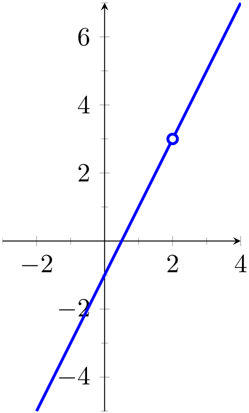
$$f(x)=\frac{2x^2-5x+2}{x-2}.$$

Since $f(x)$ is not defined at $x=2$, f is discontinuous at 2.

However,

$$\lim_{x\to 2}\frac{2x^2-5x+2}{x-2}=\lim_{x\to 2}\frac{(x-2)(2x-1)}{x-2}=\lim_{x\to 2}(2x-1)=3$$

Therefore, $f(x)$ has a removable discontinuity at 2.

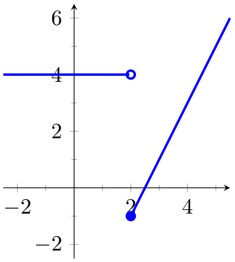


Jump Discontinuity

Consider $f(x) = \begin{cases} 4 & \text{if } x < 2 \\ 2x - 5 & \text{if } x \geq 2 \end{cases}$ at $x = 2$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 4 = 4$ and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (2x-5) = -1$

f has a jump discontinuity at 2. Also,
$f(2) = -1 = \lim_{x\to 2^+} f(x)$



One-Sided Continuity

A function $f(x)$ is called:

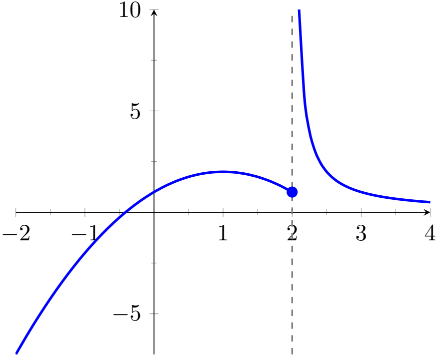
- ***Left-continuous*** *at $x = a$ if $\lim_{x\to a^-} f(x) = f(a)$*
- ***Right-continuous*** *at $x = a$ if $\lim_{x\to a^+} f(x) = f(a)$*

f is right-continuous at 2, but not left-continuous.

Infinite Discontinuity

Consider

$$f(x) = \begin{cases} -x^2 + 2x + 1 & \text{if } x \le 2 \\ \frac{1}{x-2} & \text{if } x > 2 \end{cases} \quad \text{at } x = 2$$



$f(x)$ is defined at $x = 2$, $f(2) = -2^2 + 2 \cdot 2 + 1 = 1$.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (-x^2 + 2x + 1) = 1$$

So, f is left-continuous at 2.

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{1}{x-2} = \infty$$

Therefore, $f(x)$ has an infinite discontinuity at 2.

Continuity of Basic Functions

A function f is continuous on an interval if it is continuous at every number in the interval. (We understand continuous at the endpoint to mean continuous from the right or continuous from the left.)

Theorem

The following types of functions are continuous on their domains:

- *polynomials*
- *rational functions*
- *root functions*
- *trigonometric functions*

Continuity of Basic Functions

Laws of Continuity

If $f(x)$ and $g(x)$ are continuous at $x = a$, then the following functions are also continuous at a:

- $f(x) + g(x)$ *and* $f(x) - g(x)$
- $kf(x)$ *for any constant* k
- $f(x)g(x)$
- $f(x)/g(x)$ *if* $g(a) \neq 0$

Continuity of Composite Functions

If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.

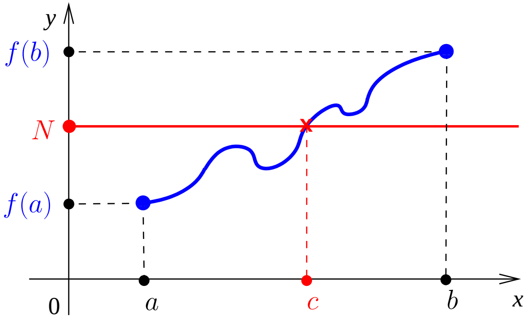
Determine where each function is continuous

1. $f(x) = 15x^{58} - 71x^{11} + 5$ is a polynomial, so it is continuous on $(-\infty, \infty)$
2. $f(x) = \sqrt{x} - \frac{x^3-4x+7}{x^2-1}$ is a difference of a root and a rational function, so it is continuous on its domain. Domain: $x \geq 0$, $x \neq \pm 1$.
   Thus, f is continuous on $[0, 1)$ and $(1, \infty)$.
3. $f(x) = \sin(\sqrt{7 - x^2})$ is a composition of a trigonometric, root and polynomial functions, so it is continuous on its domain. Domain: $7 - x^2 \geq 0$.
   Thus, f is continuous on $[-\sqrt{7}, \sqrt{7}]$
4. $f(x) = \begin{cases} x^2 - 8 & \text{if } x \leq 3 \\ \frac{1}{x-2} & \text{if } x > 3 \end{cases}$ For $x < 3$ f is continuous as a polynomial,
   for $x > 3$ f is continuous as a rational function, $x = 3$?
   $f(3) = 1 = \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 - 8) = 1 = \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \frac{1}{x-2} = 1$
   Thus, f is continuous on $(-\infty, \infty)$.

The Intermediate Value Theorem

The Intermediate Value Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



Example

Use the Intermediate Value Theorem to show that there is a root of the equation $x = \cos x$ in the interval $(0, 2)$.

Solution: Let $f(x) = x - \cos x$.
Then $f(x)$ is continuous on $[0, 2]$ as a difference of a trigonometric function and a polynomial.
$f(0) = 0 - \cos 0 = -1$, $f(2) = 2 - \cos 2 > 0$, and thus $f(0) < 0 < f(2)$.
We apply the intermediate value theorem with $a = 0$, $b = 2$ and $N = 0$.
Then there is a number c in $(0, 2)$ such that $f(c) = 0$.

In other words, there is c in $(0, 2)$ such that $c - \cos c = 0$. So c is the root of the equation $x = \cos x$.

THE END