Curve Sketching

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Asymptotes

Asymptotes provide information about the behavior of curves *in the large*.
An asymptote is a line which is tangent to a curve at infinity.

- To find horizontal asymptotes compute $\lim_{x\to\pm\infty} f(x)$. If

$$\lim_{x\to\pm\infty} f(x) = b$$

  then the line $y = b$ is a horizontal asymptote.

- To find vertical asymptotes look for values $x = a$ where $f(x)$ is undefined. If

$$\lim_{x\to\pm a} f(x) = \pm\infty$$

  then the line $x = a$ is a vertical asymptote.

First Derivative

The first derivative provides information about increasing/decreasing behavior and local maxima/minima of $f(x)$.

1. Compute $f'(x)$.
2. Find critical numbers by collecting points where $f'(x) = 0$ and points where $f'(x)$ is undefined.
3. Use the number line to determine the sign of f' between the critical points. If $f' > 0$ then f is increasing, if $f' < 0$ then f is decreasing.
4. Use the First Derivative Test to find local maxima and minima. If f' changes sign from positive to negative at $x = c$, then f has a local maximum at c. If f' changes sign from negative to positive at $x = c$, then f has a local minimum at c.

Second Derivative

The second derivative provides information about concavity and inflection points.

1. Compute $f''(x)$.
2. Collect the transition points where $f''(x) = 0$ and where $f''(x)$ is undefined.
3. Use the number line to determine the sign of f'' between the transition points. If $f'' > 0$ then f is concave upward, if $f'' < 0$ then f is concave downward.
4. Determine the inflection points, which are the points where the concavity changes.

Additional information

Domain. If $f(x)$ involves an even radical, the domain of f must be computed explicitly.

Symmetry. If f is even, the graph is symmetric about the y-axis. If f is odd, the graph is symmetric about the origin.

Intercepts. To find where the curve intercepts the y-axis we compute $f(0)$. To find where the curve intercepts x-axis we set $f(x) = 0$ and solve for x. The last step should be omitted if the equation is difficult to solve.

Periodicity. If $f(x)$ is a trigonometric function, determine the period of f.

Example: Graph $f(x) = \dfrac{x}{(x-1)^2}$

Asymptotes

To find **horizontal asymptotes** we compute

$$\lim_{x\to\pm\infty} \frac{x}{(x-1)^2} = \lim_{x\to\pm\infty} \frac{x}{x^2-2x+1} = \lim_{x\to\pm\infty} \frac{\overset{\to 0}{\frac{1}{x}}}{1 - \underset{\to 0}{\frac{2}{x}} + \underset{\to 0}{\frac{1}{x^2}}} = \frac{0}{1} = 0$$

Thus, $y = 0$ is a horizontal asymptote.

Example: Graph $f(x) = \dfrac{x}{(x-1)^2}$

Asymptotes

To find **vertical asymptotes** we find zeros of the denominator:

$(x-1)^2 = 0 \quad \Longrightarrow \quad x = 1.$

Since the numerator is not zero at $x = 1$, $f(x)$ has infinite discontinuity at $x = 1$. Thus, $x = 1$ is a vertical asymptote. We can also check this rigourously.

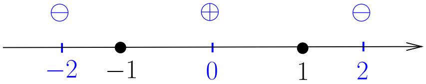
$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^{\nearrow 1}}{(x-1)^{2} \nearrow 0+} = \infty$$

Example: Graph $f(x) = \dfrac{x}{(x-1)^2}$

First Derivative

$$f'(x) = \frac{(x-1)^2 - x \cdot 2(x-1)}{(x-1)^4} = -\frac{x+1}{(x-1)^3}$$

Critical numbers: $x = -1$ (zero of the numerator) and $x = 1$ (zero of the denominator)



$$f'(-2) = -\frac{1-2}{(-2-1)^3} = -\frac{-1}{-27} < 0$$

$$f'(0) = -\frac{1+0}{(0-1)^3} = 1 > 0$$

$$f'(2) = -\frac{1+2}{(2-1)^3} = -3 < 0$$

Example: Graph $f(x) = \dfrac{x}{(x-1)^2}$

First Derivative

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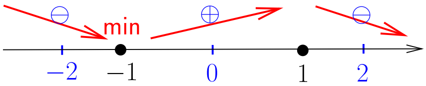


$f(x)$ is decreasing on $(-\infty, -1), (1, \infty)$,
$f(x)$ is increasing on $(-1, 1)$

Note that $f(-1) = \frac{-1}{(-1-1)^2} = -\frac{1}{4}$.

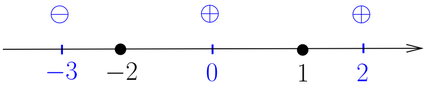
f has loc min at $(-1, -\frac{1}{4})$ and no loc max

Example: Graph $f(x) = \dfrac{x}{(x-1)^2}$

Second Derivative

$$f''(x) = \left[-\frac{1+x}{(x-1)^3}\right]' = -\frac{(x-1)^3 - 3(x-1)^2(1+x)}{(x-1)^6} = \frac{2x+4}{(x-1)^4}$$

Transition numbers: $x = -2$ (zero of the numerator) and $x = 1$(zero of the denominator)



$$f''(-3) = \frac{2(-3)+4}{(-3-1)^4} = \frac{-2}{4^4} < 0$$

$$f''(0) = \frac{0+4}{(0-1)^4} = 4 > 0$$

$$f''(2) = \frac{2 \cdot 2+4}{(2-1)^4} = 8 > 0$$

Example: Graph $f(x) = \dfrac{x}{(x-1)^2}$

Second Derivative

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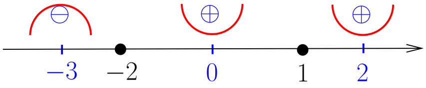


$f(x)$ is concave down on $(-\infty, -2)$,
$f(x)$ is concave up on $(-2, 1), (1, \infty)$

Note that $f(-2) = \frac{-2}{(-2-1)^2} = -\frac{2}{9}$.

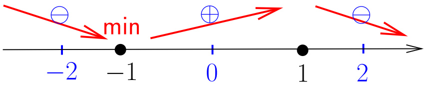
$(-2, -\frac{2}{9})$ is an inflection point

Example: Graph $f(x) = \dfrac{x}{(x-1)^2}$

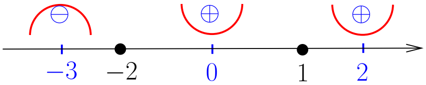
$y = 0$ is a horizontal asymptote at ∞ and $-\infty$.
$x = 1$ is a vertical asymptote

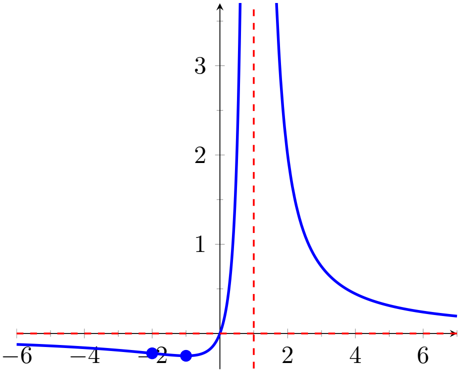
$(-1, -\frac{1}{4})$ is a loc min, $(-2, -\frac{2}{9})$ is an inflection pt.

Intervals of increase/decrease:



Concavity:





Summary

To sketch a graph of $y = f(x)$

- Determine whether f has vertical or horizontal asymptotes.
- Compute f' and determine intervals of increase/decrease and local maxima/minima.
- Compute f'' and determine intervals of concavity and inflection points.
- See if an additional information about the domain, symmetry, intercepts or periodicity is useful.
- Draw the graph

THE END