Derivatives and Rates of Change

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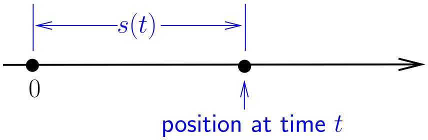
Lesson Plan

- The velocity problem GO
- Tangent Lines GO
- Rates of change GO

Velocity

An object is travelling in a straight line.

Denote by $s(t)$ the position of the object at time t.
$s(t)$ is called the position function of the object.



$$\text{Average velocity} = \frac{\text{change in position}}{\text{length of time interval}}$$

How to find the instantaneous velocity at $t = a$?

Average velocity over a very small time interval is very close to instantaneous velocity

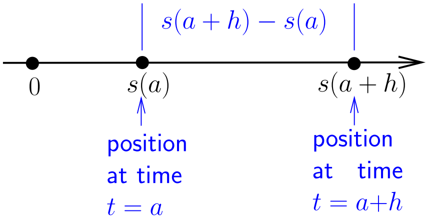
Velocity

Compute the average velocity over the time interval of length h starting at $t = a$.

Average velocity over $[a, a+h] = \dfrac{s(a+h) - s(a)}{h}$

The instantaneous velocity at $t = a$

$$v(a) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$



Example

A stone is dropped from the hight of 100 miters and falls to earth. What is the velocity of the stone after one second?

Galileo's Law: Distance traveled by a falling object is directly proportional to the square of the time it takes to fall. If the time is measured in seconds and the distance in meters, then the distance travelled after t seconds is $s(t) = 4.9t^2$ m.

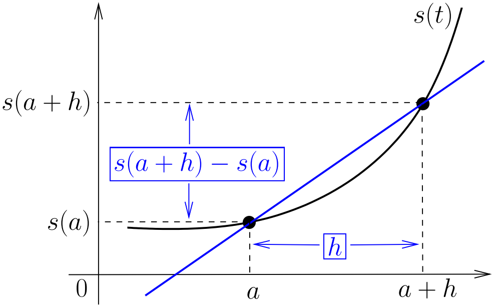
$$v(1) = \lim_{h\to 0} \frac{s(1+h) - s(1)}{h} = \lim_{h\to 0} \frac{4.9(1+h)^2 - 4.9(1)^2}{h}$$

$$= 4.9 \lim_{h\to 0} \frac{\cancel{1} + 2h + h^2 - \cancel{1}}{h} = 4.9 \lim_{h\to 0} \frac{\cancel{h}(2+h)}{\cancel{h}}$$

$$= 4.9 \lim_{h\to 0} (2+h) = \boxed{9.8\text{m/s}}$$

Graphical Interpretation of Velocity

A secant line is a line through two points on a curve.



$$\text{Slope of the secant line} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{s(a+h)-s(a)}{h}$$

Average velocity over $[a, a+h]$ is equal to the slope of the secant line.

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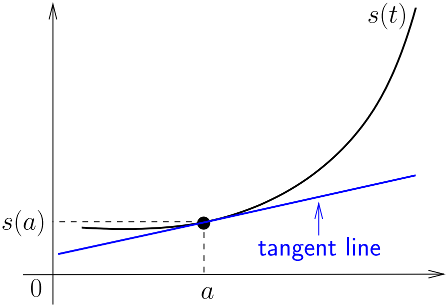

$$\text{Slope of the secant line} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{s(a+h)-s(a)}{h}$$

Average velocity over $[a, a+h]$ is equal to the slope of the secant line.

$$\text{Slope of the tangent line} = \lim_{h\to 0} \frac{s(a+h)-s(a)}{h}$$

Instantaneous velocity at $t = a$ is equal to the slope of the tangent line at $t = a$.

Tangent Lines

The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope $m = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$

Example: Find an equation of the tangent line to the graph of $f(x) = \dfrac{2}{x}$ at $(-2, -1)$.

An equation of the line with slope m through point $(-2, -1)$ is $y + 1 = m(x + 2)$.

$$m = \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h\to 0} \frac{\frac{2}{-2+h} - 1}{h} = \lim_{h\to 0} \frac{\cancel{2} - \cancel{2} + h}{h(-2+h)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(-2+h)} = \lim_{h\to 0} \frac{1}{-2+h} = -\frac{1}{2}$$

The equation of the tangent line is $y + 1 = -\frac{1}{2}(x + 2)$ or $\boxed{y = -\frac{1}{2}x - 2}$

Rates of Change

Suppose a quantity y depends on another quantity x, $y = f(x)$.
If x changes from x_1 to x_2, then y changes from $y_1 = f(x_1)$ to $y_2 = f(x_2)$.

The change in x is $\Delta x = x_2 - x_1$
The change in y is $\Delta y = y_2 - y_1 = f(x_2) - f(x_1)$

The average rate of change of y with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The instantaneous rate of change of y with respect to x at $x = x_1$ is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Denote $h = x_2 - x_1$ and $a = x_1$. Then $x_2 = x_1 + h = a + h$ and $h \to 0 (x_2 \to x_1)$

Definition of the Derivative

Definition

The derivative of a function $f(x)$ at $x = a$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

When the limit exists, we say that f is differentiable at a, otherwise f is not differentiable at a.
An equivalent definition of the derivative is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Example

Use the definition of the derivative to find $f'(a)$ for $f(x) = \sqrt{3x+1}$. Then use the answer to find the equation of the tangent line at $(1, 2)$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3(a+h)+1} - \sqrt{3a+1}}{h} \cdot \frac{\sqrt{3(a+h)+1} + \sqrt{3a+1}}{\sqrt{3(a+h)+1} + \sqrt{3a+1}}$$

$$= \lim_{h\to 0} \frac{[3(a+h)+1] - [3a+1]}{h(\sqrt{3(a+h)+1} + \sqrt{3a+1})} = \lim_{h\to 0} \frac{\cancel{3a} + 3h + \cancel{1} - \cancel{3a} - \cancel{1}}{h(\sqrt{3(a+h)+1} + \sqrt{3a+1})}$$

$$= \lim_{h\to 0} \frac{3\cancel{h}}{\cancel{h}(\sqrt{3(a+h)+1} + \sqrt{3a+1})} = \lim_{h\to 0} \frac{3}{\sqrt{3(a+h)+1} + \sqrt{3a+1}} = \frac{3}{2\sqrt{3a+1}}$$

Example

Use the definition of the derivative to find $f'(a)$ for $f(x) = \sqrt{3x+1}$. Then use the answer to find the equation of the tangent line at $(1, 2)$

Therefore, $\boxed{f'(a) = \dfrac{3}{2\sqrt{3a+1}}}$

The slope of the tangent line at $(1, 2)$ is

$$m = f'(1) = \frac{3}{2\sqrt{3 \cdot 1 + 1}} = \frac{3}{4}$$

An equation of the tangent line is $y - 2 = \dfrac{3}{4}(x - 1)$ or $\boxed{y = \dfrac{3}{4}x + \dfrac{5}{4}}$

Summary

- The derivative of $f(x)$ at $x = a$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

  If the limit exists, we say that f is differentiable at a.

- The tangent line to the curve $y = f(x)$ at point $P(a, f(a))$ is the line through P with slope $f'(a)$. The equation of the tangent line in point-slope form is

$$y - f(a) = f'(a)(x - a)$$

- The (instantaneous) velocity of an object travelling in a straight line is $s'(t)$, where $s(t)$ is the position of the object at time t.

THE END