Derivatives and the Shapes of Graphs

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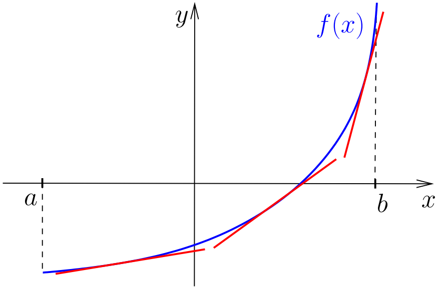
Lesson Plan

- What does f' say about f?
- What does f'' say about f?

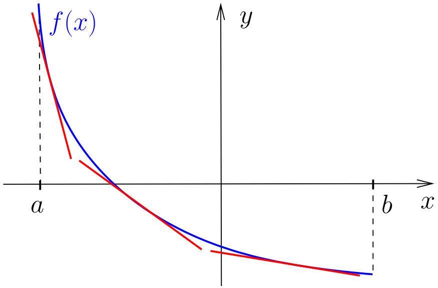
Increasing/Decreasing Behavior of Functions

Definition

- $f(x)$ is *increasing* on (a,b) if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in (a,b).
- $f(x)$ is *decreasing* on (a,b) if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in (a,b).



Increasing function.
Tangent lines have positive slope.



Decreasing function.
Tangent lines have negative slope.

Increasing/Decreasing Behavior of Functions

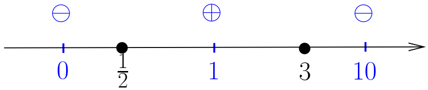
Increasing/Decreasing Test

- *If $f'(x) > 0$ on an interval, then f is increasing on that interval.*
- *If $f'(x) < 0$ on an interval, then f is decreasing on that interval.*

Example. Determine where $f(x) = -\frac{2}{3}x^3 + \frac{7}{2}x^2 - 3x + 1$ is increasing and where it is decreasing.

$f'(x) = -2x^2 + 7x - 3$. Since f' is continuous, it can change sign only at zero. We set

$$-2x^2 + 7x - 3 = 0 \quad \Longrightarrow \quad x_{1,2} = \frac{-7 \pm \sqrt{49 - 24}}{-4} = \frac{-7 \pm 5}{-4} \quad \text{or} \quad x_1 = \frac{1}{2},\ x_2 = 3$$



$$f'(0) = -2(0)^2 + 7(0) - 3 = -3 < 0$$
$$f'(1) = -2(1)^2 + 7(1) - 3 = 2 > 0$$
$$f'(1) = -2(10)^2 + 7(10) - 3 = -133 < 0$$

Increasing/Decreasing Behavior of Functions

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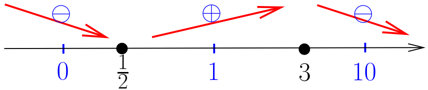
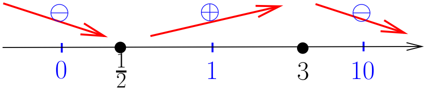
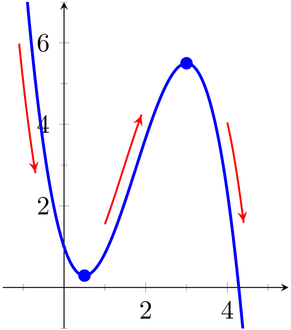


$f(x)$ is increasing on $(\frac{1}{2}, 3)$,
$f(x)$ is decreasing on $(-\infty, \frac{1}{2}), (3, \infty)$

Increasing/Decreasing Behavior of Functions

Graph $f(x) = -\frac{2}{3}x^3 + \frac{7}{2}x^2 - 3x + 1$.





- $f(x)$ is decreasing on $(-\infty, \frac{1}{2})$. Note that $f(\frac{1}{2}) = \frac{7}{24}$
- $f(x)$ is increasing on $(\frac{1}{2}, 3)$. Note that $f(3) = 5.5$
- $f(x)$ is decreasing on $(3, \infty)$.

$f(x)$ has local minimum at $x = \frac{1}{2}$.

$f(x)$ has local maximum at $x = 3$.

Increasing/Decreasing Behavior of Functions

First Derivative Test for Extrema

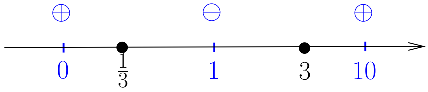
Suppose that is c a critical number of a continuous function f.

- *If f' changes sign from positive to negative at c, then f has a* local maximum *at c.*
- *If f' changes sign from negative to positive at c, then f has a* local minimum *at c.*
- *If f' does not change sign at c, then f has no local maximum or minimum at c.*

Example. Find local minimum and maximum values of $f(x) = x^3 - 5x^2 + 3x - 1$.

$f'(x) = 3x^2 - 10x - 3$. Since f' is continuous, it can change sign only at zero. We set

$$3x^2 - 10x - 3 = 0 \quad \Longrightarrow \quad x_{1,2} = \frac{10 \pm \sqrt{100-36}}{6} = \frac{10 \pm 8}{6} \quad \text{or} \quad x_1 = \frac{1}{3}, \; x_2 = 3$$



$$f'(0) = 3(0)^2 - 10(0) + 3 = 3 > 0$$
$$f'(1) = 3(1)^2 - 10(1) + 3 = -4 < 0$$
$$f'(10) = 3(10)^2 - 10(10) + 3 = 203 > 0$$

Increasing/Decreasing Behavior of Functions

First Derivative Test for Extrema

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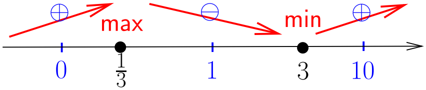
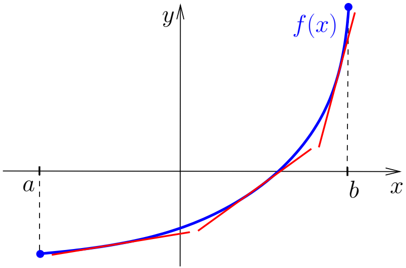


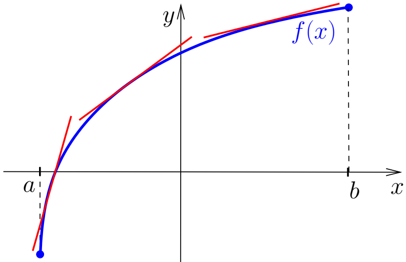
local minimum is at $(3, -10)$,
local maximum is at $\left(\frac{1}{3}, -\frac{14}{27}\right)$

Concavity

Suppose we know that f is increasing on (a, b). We can draw its graph in two ways.



Graph bends upward
Function is concave up



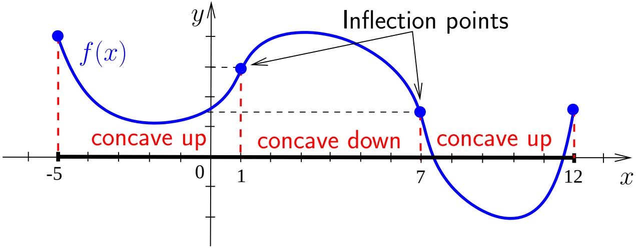
Graph bends downward
Function is concave down

Definition

- *If the graph of f lies above all of its tangents on (a, b), then f is concave up.*
- *If the graph of f lies below all of its tangents on (a, b), then f is concave down.*

Example

Where is the function concave up/down?



f is concave up on $(-5, 1)$

f is concave down on $(1, 7)$

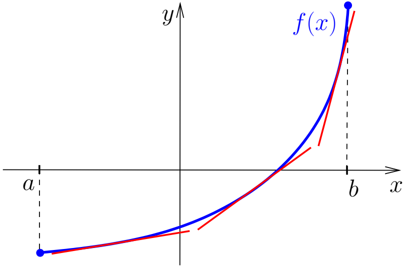
f is concave up on $(7, 12)$

Inflection points are $(1, 3), (7, 1.5)$

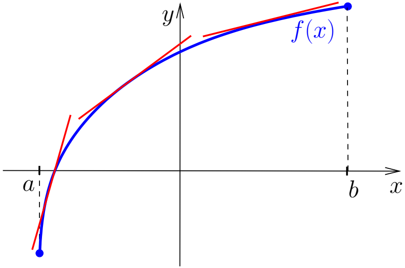
Definition

A point P on a curve $y = f(x)$ is called an <u>*inflection point*</u> *if f is continuous there and the curve changes from concave up to concave down or from concave down to concave up at P.*

Second Derivative Test for Concavity



Function is concave up
Slopes of tangent lines are increasing



Function is concave down
Slopes of tangent lines are decreasing

Second Derivative Test for Concavity

- *If* $f''(x) > 0$ *on* (a, b), *then* f *is concave up on* (a, b).
- *If* $f''(x) < 0$ *on* (a, b), *then* f *is concave down on* (a, b).

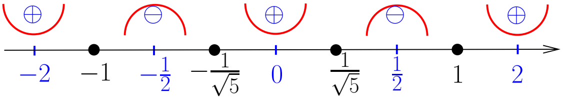
Example

Find the intervals of concavity and the inflection points for $f(x) = (x^2-1)^3$.

$$f'(x) = 3(x^2-1)^2 \cdot 2x = 6x(x^2-1)^2$$

$$f''(x) = 6(x^2-1)^2 + 6x \cdot 2(x^2-1) \cdot 2x = 6(x^2-1)(x^2-1+2x\cdot 2x) = 6(x^2-1)(5x^2-1)$$

$$6(x^2-1)(5x^2-1) = 0 \quad \Leftrightarrow \quad x^2-1=0 \text{ or } 5x^2-1=0 \quad \Leftrightarrow \quad x=\pm 1 \text{ or } x = \pm\frac{1}{\sqrt{5}}$$



$$f''(-2) = 6((-2)^2-1)(5(-2)^2-1) = 6(3)(19) > 0$$

$$f''(-\tfrac{1}{2}) = 6((-\tfrac{1}{2})^2-1)(5(-\tfrac{1}{2})^2-1) = 6(-\tfrac{3}{4})(\tfrac{1}{4}) < 0$$

$$f''(0) = 6(-1)(-1) > 0$$

$$f''(\tfrac{1}{2}) = f''(-\tfrac{1}{2}) = 6(-\tfrac{3}{4})(\tfrac{1}{4}) < 0$$

$$f''(2) = f''(-2) = 6(3)(19) > 0$$

f is concave up on $(-\infty,-1), (-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}), (1,\infty)$

f is concave down on $(-1,-\frac{1}{\sqrt{5}}), (\frac{1}{\sqrt{5}}, 1)$

Inflection points are $(\pm 1, 0), (\pm\frac{1}{\sqrt{5}}, (\frac{4}{5})^3)$

Second Derivative Test for Extrema

Second Derivative Test for Extrema

Suppose c is a critical number of f and $f''(c)$ exists.

- *If $f''(c) < 0$, then f has a local maximum at c.*
- *If $f''(c) > 0$, then f has a local minimum at c.*
- *If $f''(c) = 0$, then the test is inconclusive.*

Example. Use the second derivative test to find local extrema for $f(x) = x^4 - 8x^2 + 1$.

First we find critical numbers.

$f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 0 \implies x = 0$ or $x = \pm 2$.

$f''(x) = 12x^2 - 16$. Substitute critical numbers into $f''(x)$:

$f''(0) = 12 \cdot 0^2 - 16 = -16 < 0 \implies f$ has local max at $x = 0$

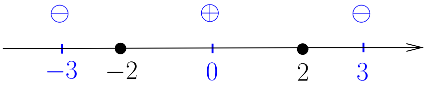
$f''(-2) = 12(-2)^2 - 16 = 8 > 0$ and $f''(2) = 8 > 0 \Rightarrow f$ has local min at $x = \pm 2$

Example

Find where $f(x) = \dfrac{x}{x^2+4}$ is increasing/decreasing, local extrema, intervals of concavity and inflection points.

Increasing/decreasing behavior. $f'(x) = \dfrac{1(x^2+4) - x \cdot 2x}{(x^2+4)^2} = \dfrac{4-x^2}{(x^2+4)^2}$.

Since the denominator $x^2 + 4 \neq 0$, critical numbers are zeros of the numerator. We set $4 - x^2 = 0$ and obtain that $x = \pm 2$ are critical numbers.



$$f'(-3) = \frac{4-(-3)^2}{((-3)^2+4)^2} = \frac{-5}{13^2} < 0$$

$$f'(0) = \frac{4-0^2}{(0^2+4)^2} = \frac{1}{4} > 0$$

$$f'(3) = \frac{4-(3)^2}{((3)^2+4)^2} = \frac{-5}{13^2} < 0$$

Example

Find where $f(x) = \dfrac{x}{x^2+4}$ is increasing/decreasing, local extrema, intervals of concavity and inflection points.

Increasing/decreasing behavior. $f'(x) = \dfrac{1(x^2+4) - x \cdot 2x}{(x^2+4)^2} = \dfrac{4-x^2}{(x^2+4)^2}$.

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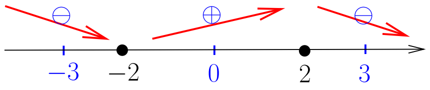


$f(x)$ is increasing on $(-\infty, -2), (2, \infty)$,
$f(x)$ is decreasing on $(-2, 2)$

Note that $f(-2) = \frac{-2}{(-2)^2+4} = -\frac{1}{4}$ and $f(2) = \frac{1}{4}$.

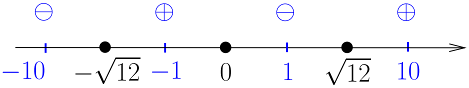
f has loc min at $(-2, -\frac{1}{4})$ and loc max at $(2, \frac{1}{4})$

Example

Find where $f(x) = \dfrac{x}{x^2+4}$ is increasing/decreasing, its local extrema, intervals of concavity and inflection points.

Concavity. Recall that $f'(x) = \dfrac{4-x^2}{(x^2+4)^2}$.

$$f''(x) = \frac{-2x(x^2+4)^2 - (4-x^2)2(x^2+4)2x}{(x^2+4)^4} = \frac{2x(x^2-12)}{(x^2+4)^3} = 0 \Rightarrow x = 0, x = \pm\sqrt{12}$$



$$f''(-10) = \frac{2(-10)((-10)^2-12)}{((-10)^2+4)^3} = \frac{-20 \cdot 88}{(104)^3} < 0$$

$$f''(-1) = \frac{2(-1)((-1)^2-12)}{((-1)^2+4)^3} = \frac{22}{(5)^3} > 0$$

$$f''(1) = \frac{2(1)(1^2-12)}{((1)^2+4)^3} = -\frac{22}{(5)^3} < 0$$

$$f''(10) = \frac{2(10)((10)^2-12)}{((10)^2+4)^3} = \frac{20 \cdot 88}{(104)^3} > 0$$

Example

Find where $f(x) = \dfrac{x}{x^2+4}$ is increasing/decreasing, its local extrema, intervals of concavity and inflection points.

Concavity. Recall that $f'(x) = \dfrac{4-x^2}{(x^2+4)^2}$.

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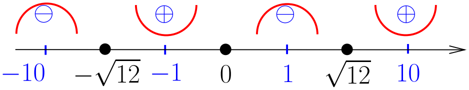

$f(x)$ is concave down on $(-\infty, -\sqrt{12}), (0, \sqrt{12})$,
$f(x)$ is concave up on $(-\sqrt{12}, 0), (\sqrt{12}, \infty)$.
Inflection points are
$(-\sqrt{12}, -\frac{\sqrt{12}}{16})$, $(0, 0)$, $(\sqrt{12}, \frac{\sqrt{12}}{16})$.

Summary

The signs of the first two derivatives provide the following information:

- *If* $f'(x) > 0$ *on* (a, b) *then* f *is increasing on* (a, b).
- *If* $f'(x) < 0$ *on* (a, b) *then* f *is decreasing on* (a, b).
- *If* $f'(x)$ *changes sign from* $+$ *to* $-$ *at* $x = c$ *then* $(c, f(c))$ *is a local max.*
- *If* $f'(x)$ *changes sign from* $-$ *to* $+$ *at* $x = c$ *then* $(c, f(c))$ *is a local min.*
- *If* $f''(x) > 0$ *on* (a, b) *then* f *is concave up on* (a, b).
- *If* $f''(x) < 0$ *on* (a, b) *then* f *is concave down on* (a, b).
- *If* $f''(x)$ *changes sign at* $x = c$ *then* $(c, f(c))$ *is an inflection point.*

THE END