Evaluating Limits

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Lesson Plan

- Strategy to calculate limits of rational and irrational functions.
- The Squeeze Theorem
- Trigonometric limits

Evaluating Limits

1. $$\lim_{x\to 1}\frac{x^2-4}{x^2-5x+6}=\frac{1^2-4}{1^2-5\cdot 1+6}=\boxed{-\frac{3}{2}}$$ Use limit laws to substitute $x=1$.

2. $$\lim_{x\to 2}\frac{x^2-4}{x^2-5x+6}\cancel{=\frac{2^2-4}{2^2-5\cdot 2+6}=\frac{0}{0}}$$ Cannot use limit laws to substitute $x=2$!

$$=\lim_{x\to 2}\frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(x-3)}=\lim_{x\to 2}\frac{x+2}{x-3}=\frac{2+2}{2-3}=\boxed{-4}$$

3. $$\lim_{x\to 3}\frac{x^2-4}{x^2-5x+6}\cancel{=\frac{3^2-4}{3^2-5\cdot 3+6}=\frac{5}{0}}$$ Cannot use limit laws to substitute $x=3$!

$$\lim_{x\to 3^-}\frac{x^2-4 \to 5}{(x-2)\to 1\,(x-3)\to 0^-}=-\infty \neq \lim_{x\to 3^+}\frac{x^2-4 \to 5}{(x-2)\to 1\,(x-3)\to 0^+}=\infty \qquad \boxed{\text{DNE}}$$

Strategy

To compute a limit of a fraction when x approaches a number plug in the limiting value of x into the numerator and the denominator.

- If the denominator is not zero, divide the numerator by the denominator and obtain the answer.
- If the denominator is zero but the numerator is not zero, consider one-sided limits to select the best answer from $-\infty$, ∞ and DNE.
- If both the numerator and the denominator are zeros, transform algebraically and cancel common factors in the numerator and the denominator

Example 1

Evaluate $\lim_{x\to 0} \frac{(3-x)^2-9}{x}$

1. Try to substitute $x=0$: $\frac{(3-0)^2-9}{0} = \frac{0}{0}$
2. Transform algebraically and cancel:

$$\lim_{x\to 0} \frac{(3-x)^2-9}{x} = \lim_{x\to 0} \frac{\cancel{9}-6x+x^2-\cancel{9}}{x} = \lim_{x\to 0} \frac{\cancel{x}(-6+x)}{\cancel{x}} = \lim_{x\to 0}(-6+x)$$

3. Substitute $x=0$: $\lim_{x\to 0}(-6+x) = -6+0 = \boxed{-6}$

Example 2

Evaluate $\lim_{x\to 1} \dfrac{x^2+x-2}{x^2-3x+2}$

1. Try to substitute $x=1$: $\dfrac{1^2+1-2}{1^2-3\cdot 1+2} = \dfrac{0}{0}$
2. Transform algebraically and cancel:

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-3x+2} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+2)}{\cancel{(x-1)}(x-2)} = \lim_{x\to 1} \frac{x+2}{x-2}$$

3. Substitute $x=1$: $\lim_{x\to 1} \dfrac{x+2}{x-2} = \dfrac{1+2}{1-2} = \boxed{-3}$

Example 3 *Multiplying by the Conjugate*

Evaluate $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$

1. Try to substitute $x = 4$: $\frac{\sqrt{4}-2}{4-4} = \frac{0}{0}$
2. Transform algebraically and cancel:

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to 4} \frac{\cancel{x-4}}{\cancel{(x-4)}(\sqrt{x}+2)} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2}$$

3. Substitute $x = 4$: $\lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{\sqrt{4}+2} = \boxed{\frac{1}{4}}$

Example 4 *Multiplying by the Conjugate*

Evaluate $\displaystyle\lim_{x\to 5}\frac{x-5}{\sqrt{x+4}-3}$

1. Try to substitute $x=5$: $\displaystyle\frac{5-5}{\sqrt{5+4}-3}=\frac{0}{0}$
2. Transform algebraically and cancel:

$$\lim_{x\to 5}\frac{x-5}{\sqrt{x+4}-3}\cdot\frac{\sqrt{x+4}+3}{\sqrt{x+4}+3}=\lim_{x\to 5}\frac{(x-5)(\sqrt{x+4}+3)}{(\sqrt{x+4})^2-3^2}=\lim_{x\to 5}\frac{\cancel{(x-5)}(\sqrt{x+4}+3)}{\cancel{x-5}}$$

3. Substitute $x=5$: $\displaystyle\lim_{x\to 5}(\sqrt{x+4}+3)=\sqrt{5+4}+3=\boxed{6}$

Example 5

Evaluate $\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right)$

1. Try to substitute $x=1$: $\frac{1}{1-1}-\frac{2}{1^2-1}=\frac{1}{0}-\frac{2}{0}$
2. Transform algebraically and cancel:

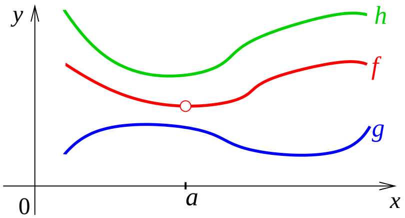
$$\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right)=\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{(x-1)(x+1)}\right)$$

$$=\lim_{x\to 1}\left(\frac{x+1}{(x-1)(x+1)}-\frac{2}{(x-1)(x+1)}\right)=\lim_{x\to 1}\frac{\cancel{x-1}}{\cancel{(x-1)}(x+1)}$$

3. Substitute $x=1$: $\lim_{x\to 1}\frac{1}{x+1}=\frac{1}{1+1}=\boxed{\frac{1}{2}}$

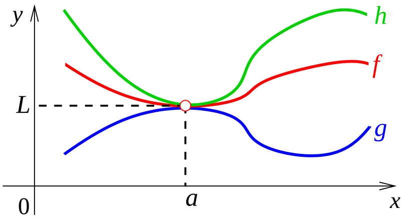
The Squeeze Theorem

$f(x)$ is trapped between $g(x)$ and $h(x)$



$g(x) \leq f(x) \leq h(x)$ for all x near a.

$f(x)$ is squeezed at $x{=}a$ by $g(x)$ and $h(x)$



$g(x) \leq f(x) \leq h(x)$ and $\lim\limits_{x\to a} g(x){=}\lim\limits_{x\to a} h(x){=}L$

The Squeeze Theorem

If $g(x) \leq f(x) \leq h(x)$ *when* x *is near* a *(except possibly at* a *) and* $\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a} h(x) = L$ *then* $\lim\limits_{x\to a} f(x) = L$.

Example 1

Evaluate $\displaystyle \lim_{x\to 0} x^2 \sin\frac{1}{x}$

$\displaystyle\lim_{x\to 0} \sin\frac{1}{x}$ DNE, so we cannot use the Product Law.

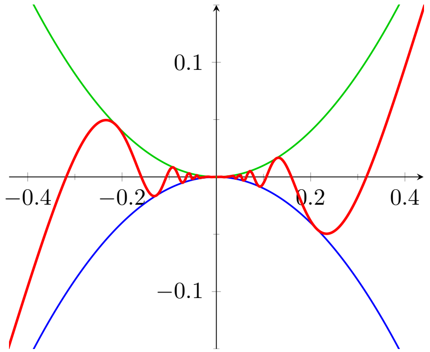
For all $x \neq 0$ $\qquad -1 \leq \sin\frac{1}{x} \leq 1$

Multiply each side of the inequalities by x^2

$$-x^2 \leq \quad x^2 \sin\frac{1}{x} \quad \leq x^2$$

$$0$$

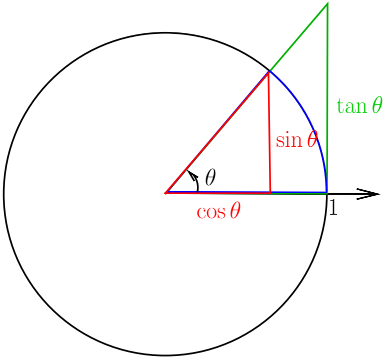
By Squeeze Theorem $\displaystyle\lim_{x\to 0} x^2 \sin\frac{1}{x} = \boxed{0}$



$g(x) = -x^2$, $f(x) = x^2 \sin\frac{1}{x}$, $h(x) = x^2$

Example 2

Show that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$



Area of the red triangle	$\leq$	Area of the sector	$\leq$	Area of the green triangle
$\frac{1}{2}\sin\theta\cos\theta$	$\leq$	$\frac{1}{2}\theta$	$\leq$	$\frac{1}{2}\tan\theta$
$\frac{\cancel{\frac{1}{2}\sin\theta}\cos\theta}{\cancel{\frac{1}{2}\sin\theta}}$	$\leq$	$\frac{\cancel{\frac{1}{2}}\theta}{\cancel{\frac{1}{2}}\sin\theta}$	$\leq$	$\frac{\cancel{\frac{1}{2}\tan\theta}^{\frac{1}{\cos\theta}}}{\cancel{\frac{1}{2}\sin\theta}}$
$\cos\theta$	$\leq$	$\frac{\theta}{\sin\theta}$	$\leq$	$\frac{1}{\cos\theta}$
		$\Downarrow$ 1		

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = \lim_{\theta\to 0} \frac{1}{\frac{\theta}{\sin\theta}} = \frac{1}{1} = 1$$

Example 1

Evaluate $\displaystyle\lim_{x\to 0}\frac{\sin 4x}{x}$

$$\lim_{x\to 0}\frac{\sin \overset{\theta}{4x}}{x}=\lim_{x\to 0}\frac{4\sin 4x}{4x}=4\underbrace{\lim_{x\to 0}\frac{\sin 4x}{4x}}_{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1}=4\cdot 1=\boxed{4}$$

Example 2

Evaluate $\lim_{x\to 0} \dfrac{\tan x}{\sin 2x}$

$$\lim_{x\to 0} \frac{\tan x}{\sin 2x} = \lim_{x\to 0} \frac{\frac{\sin x}{\cos x} \cdot \frac{1}{x}}{\sin 2x \cdot \frac{1}{x}} = \lim_{x\to 0} \frac{\overset{1}{\cancel{\frac{1}{\cos x}}} \cdot \overset{1}{\cancel{\frac{\sin x}{x}}}}{\overset{1}{\cancel{\frac{\sin 2x}{2x}}} \cdot 2} = \boxed{\frac{1}{2}}$$

Example 3

Evaluate $\displaystyle \lim_{x\to 0} \frac{\sin x}{x+\tan x}$

$$\lim_{x\to 0} \frac{\sin x}{x+\tan x} = \lim_{x\to 0} \frac{\frac{\sin x}{x}}{\frac{x+\tan x}{x}} = \lim_{x\to 0} \frac{\frac{\sin x}{x}}{1+\frac{\tan x}{x}} = \lim_{x\to 0} \frac{\cancelto{1}{\frac{\sin x}{x}}}{1+\cancelto{1}{\frac{\sin x}{x}}\cdot\cancelto{1}{\frac{1}{\cos x}}} = \frac{1}{1+1} = \boxed{\frac{1}{2}}$$

THE END