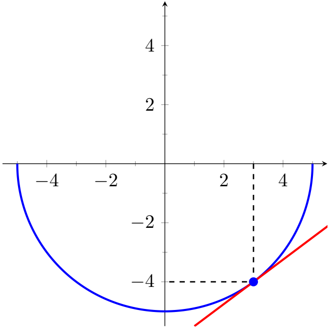
Implicit Differentiation

Tamara Kucherenko

Tangent Line to a Circle

Find an equation of the tangent line to the curve $x^2 + y^2 = 25$ at $(3, -4)$



Solution 1: $x^2 + y^2 = 25 \implies y = \pm\sqrt{25 - x^2}$

Since we are interested in the point (3,-4), we choose $y = f(x) = -\sqrt{25 - x^2}$

$$\frac{dy}{dx} = f'(x) = -\frac{1}{2\sqrt{25 - x^2}} \cdot (-2x) = \frac{x}{\sqrt{25 - x^2}}$$

The slope of the tangent line at $(3, -4)$ is

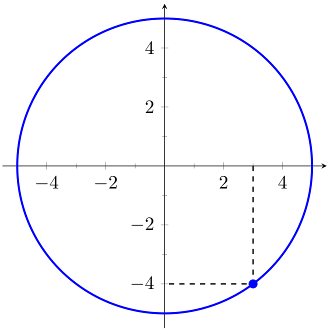
$$\frac{dy}{dx}(3) = \frac{3}{\sqrt{25 - 3^2}} = \frac{3}{4}$$

The equation of the tangent line is

$y + 4 = \frac{3}{4}(x - 3)$ or $\boxed{y = \frac{3}{4}x - \frac{25}{4}}$

Implicit Differentiation

Find an equation of the tangent line to the curve $x^2 + y^2 = 25$ at $(3, -4)$



Solution 2: Differentiate both sides of $x^2+y^2=25$

$$\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = 0 \;\Rightarrow\; 2x + \frac{d}{dx}\left(y^2\right) = 0$$

We think of y as a function $y = f(x)$.

Then $y^2 = f(x)^2$, and by the Chain Rule

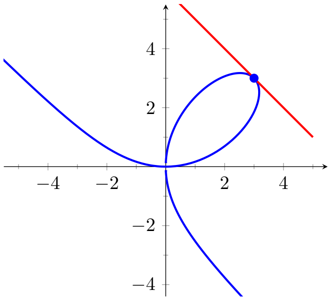
$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(f(x)^2\right) = 2f(x)\cdot f'(x) = 2y\frac{dy}{dx}$$

$$2x + 2y\frac{dy}{dx} = 0 \;\Rightarrow\; 2y\frac{dy}{dx} = -2x \;\Rightarrow\; \frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{dy}{dx}(3,-4) = -\frac{3}{-4} = \frac{3}{4} \quad\Longrightarrow\quad \boxed{y = \frac{3}{4}x - \frac{25}{4}}$$

Example 1

Find an equation of the tangent line to the curve $x^3 + y^3 = 6xy$ at $(3, 3)$



The folium of Descartes

Solution. Differentiate both sides with respect to x.

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(6xy)$$
$$3x^2 + 3y^2y' = 6y + 6xy'$$

Solve for y'.

$$x^2 + y^2y' = 2y + 2xy'$$
$$y^2y' - 2xy' = 2y - x^2$$
$$y'(y^2 - 2x) = 2y - x^2 \Rightarrow y' = \frac{2y - x^2}{y^2 - 2x}$$

Substitute $x = 3,\ y = 3$ to find the slope m of the tangent line at $(3, 3)$.

$$m = \frac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3} = -1 \Rightarrow y - 3 = -(x - 3) \Rightarrow \boxed{y = -x + 6}$$

Example 2

Find y' if $1 + \sin y = \cos(xy)$

Differentiate both sides with respect to x.

$$\frac{d}{dx}(1 + \sin y) = \frac{d}{dx}(\cos(xy))$$

$$0 + \frac{d}{dx}(\sin y) = -\sin(xy)\frac{d}{dx}(xy)$$

$$(\cos y)y' = -\sin(xy)(y + xy')$$

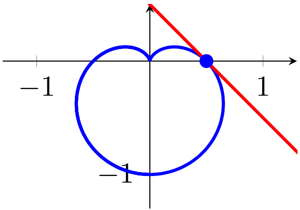
Solve for y'.

$$y' \cos y + xy' \sin(xy) = -y\sin(xy)$$

$$y'(\cos y + x\sin(xy)) = -y\sin(xy) \Rightarrow \boxed{y' = \frac{-y\sin(xy)}{\cos y + x\sin(xy)}}$$

Example 3

Use implicit differentiation to find an equation of the tangent line to the cardioid $x^2+y^2=(2x^2+2y^2+y)^2$ at $(\frac{1}{2},0)$.



Differentiate both sides with respect to x.

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}\left((2x^2+2y^2+y)^2\right)$$

$$2x+2yy'=2(2x^2+2y^2+y)(4x+4yy'+y')$$

First substitute $x=\frac{1}{2}$, $y=0$, then solve for y'.

$$2\cdot\frac{1}{2}+2\cdot 0\cdot y'=2\left(2\left(\frac{1}{2}\right)^2+2(0)^2+0\right)\left(4\cdot\frac{1}{2}+4\cdot 0\cdot y'+y'\right)$$

$$1=2+y' \Rightarrow y'=-1$$

The slope of the tangent line at $(\frac{1}{2},0)$ is -1.

The equation of the tangent line is $\boxed{y=-x+\frac{1}{2}}$

Summary

Implicit differentiation is used to compute $\frac{dy}{dx}$ when x and y are related by an equation.

1. Take the derivative of both sides of the equation with respect to x
2. Solve for y' by collecting the terms involving y' on one side and the remaining terms on the other side of the equation.

Remember to include the factor y' when differentiating expressions involving y with respect to x.

THE END