Limits Involving Infinity and Asymptotes

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Lesson Plan

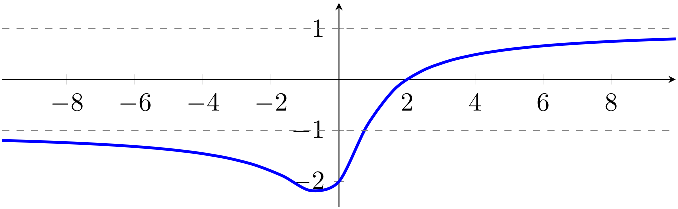
- Limits at Infinity GO
- Calculating Limits at Infinity GO
- Vertical and Horizontal Asymptotes GO

Limits at Infinity

Definition

- $\lim_{x\to\infty} f(x) = L$ *means that the values of* $f(x)$ *can be made as close to* L *as we like by taking any* x *which is sufficiently large.*
- $\lim_{x\to-\infty} f(x) = L$ *means that the values of* $f(x)$ *can be made as close to* L *as we like by taking any* x *which is sufficiently large negative.*

Example: $f(x) = \dfrac{x-2}{\sqrt{x^2+1}}$



- $\lim_{x\to\infty} f(x) = 1$
- $\lim_{x\to-\infty} f(x) = -1$

Limits at Infinity

Investigate $\lim_{x\to\infty} \frac{1}{x}$ **and** $\lim_{x\to-\infty} \frac{1}{x}$

$x > 100 \quad \rightsquigarrow \quad \frac{1}{x} < \frac{1}{100} = 0.01$

$x > 10,000 \quad \rightsquigarrow \quad \frac{1}{x} < \frac{1}{10000} = 0.0001$

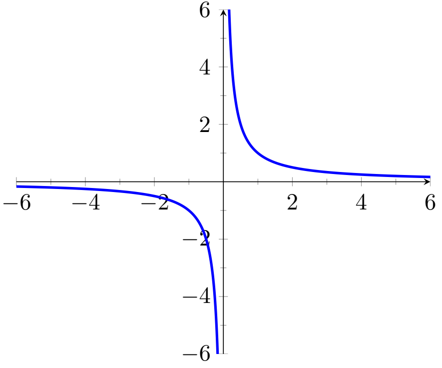
$x = 62,708 \quad \rightsquigarrow \quad \frac{1}{x} = \frac{1}{62708} \approx 0.000015947$

When x is large, $1/x$ is small. By taking x large enough we can make $1/x$ as small as we like.

$\boxed{\lim_{x\to\infty} \frac{1}{x} = 0}$. Similarly, $\boxed{\lim_{x\to-\infty} \frac{1}{x} = 0}$

because when x is large negative, $\frac{1}{x}$ is small negative.

$x = -10,000 \quad \rightsquigarrow \quad \frac{1}{x} = \frac{1}{-10000} = -0.0001$



Limits at Infinity

The Limit Laws and the Squeeze Theorem are valid when $"x \to a"$ is replaced by $"x \to \infty"$ or $"x \to -\infty"$.

Applying the Product Law repeatedly to the limits from the previous example we obtain

For any positive integer n : $\lim\limits_{x\to\infty} \frac{1}{x^n} = 0, \quad \lim\limits_{x\to-\infty} \frac{1}{x^n} = 0$

Example 1

Evaluate

$$\lim_{x\to\infty} \frac{2x+1}{x-2}$$

Solution:

$$\lim_{x\to\infty} \frac{2x+1}{x-2} = \lim_{x\to\infty} \frac{\frac{2x+1}{x}}{\frac{x-2}{x}} = \lim_{x\to\infty} \frac{2+\cancelto{0}{\frac{1}{x}}}{1-2\cancelto{0}{\frac{1}{x}}} = \boxed{2}$$

Example 2

Evaluate

$$\lim_{x\to\infty} \frac{2x^3 - 5x + 1}{4x^3 - 7x^2 + 2}$$

Solution:

$$\lim_{x\to\infty} \frac{2x^3 - 5x + 1}{4x^3 - 7x^2 + 2} = \lim_{x\to\infty} \frac{\frac{2x^3-5x+1}{x^3}}{\frac{4x^3-7x^2+2}{x^3}} = \lim_{x\to\infty} \frac{2 - 5\cancelto{0}{\frac{1}{x^2}} + \cancelto{0}{\frac{1}{x^3}}}{4 - 7\cancelto{0}{\frac{1}{x}} + 2\cancelto{0}{\frac{1}{x^3}}} = \frac{2}{4} = \boxed{\frac{1}{2}}$$

Exactly the same calculation shows that the limit as $x \to -\infty$ is also $\frac{1}{2}$.

Example 3

Evaluate

$$\lim_{x\to-\infty}\frac{x^2+4}{2x-5}$$

Solution:

$$\lim_{x\to-\infty}\frac{x^2+4}{2x-5}=\lim_{x\to-\infty}\frac{\frac{x^2+4}{x}}{\frac{2x-5}{x}}=\lim_{x\to-\infty}\frac{x+\cancelto{0}{\frac{4}{x}}}{2-\cancelto{0}{\frac{5}{x}}}=\boxed{-\infty}$$

Example 4

Evaluate

$$\lim_{x\to\infty}(x^3-2x)$$

Solution:

$$\lim_{x\to\infty}(x^3-2x)=\lim_{x\to\infty}\overset{\infty}{x}\cdot\overset{\infty}{(x^2-2)}=\boxed{\infty}$$

Example 5

Evaluate

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$$

Solution:

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$

$$=\lim_{x\to\infty}\frac{(\cancel{x^2}+1)-\cancel{x^2}}{\sqrt{x^2+1}+x}$$

$$=\lim_{x\to\infty}\frac{1}{\underbrace{\sqrt{x^2+1}}_{\infty}+\underbrace{x}_{\infty}}=\frac{1}{\infty}=\boxed{0}$$

Example 6

Evaluate $\displaystyle\lim_{x\to\infty}\frac{x-2}{\sqrt{x^2+1}}$ and $\displaystyle\lim_{x\to-\infty}\frac{x-2}{\sqrt{x^2+1}}$

Solution: $$\frac{x-2}{\sqrt{x^2+1}}=\frac{x-2}{\sqrt{x^2(1+\frac{1}{x^2})}}=\frac{x-2}{\sqrt{x^2}\cdot\sqrt{1+\frac{1}{x^2}}}=\frac{x-2}{|x|\cdot\sqrt{1+\frac{1}{x^2}}}$$

$$\lim_{x\to\infty}\frac{x-2}{|x|\cdot\sqrt{1+\frac{1}{x^2}}}=\lim_{x\to\infty}\frac{x-2}{x\cdot\sqrt{1+\frac{1}{x^2}}}=\lim_{x\to\infty}\frac{\frac{x-2}{x}}{\frac{x}{x}\cdot\sqrt{1+\frac{1}{x^2}}}=\lim_{x\to\infty}\frac{1-\cancelto{0}{\frac{2}{x}}}{\sqrt{1+\cancelto{0}{\frac{1}{x^2}}}}=\boxed{1}$$

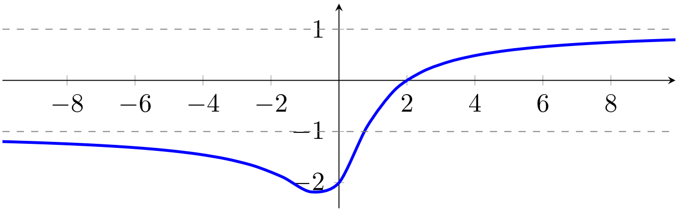
$$\lim_{x\to-\infty}\frac{x-2}{|x|\cdot\sqrt{1+\frac{1}{x^2}}}=\lim_{x\to-\infty}\frac{x-2}{-x\cdot\sqrt{1+\frac{1}{x^2}}}=\lim_{x\to-\infty}\frac{1-\cancelto{0}{\frac{2}{x}}}{-\sqrt{1+\cancelto{0}{\frac{1}{x^2}}}}=\frac{1}{-\sqrt{1}}=\boxed{-1}$$

Example 6

$$\lim_{x\to\infty} \frac{x-2}{\sqrt{x^2+1}} = 1 \quad \text{and} \quad \lim_{x\to-\infty} \frac{x-2}{\sqrt{x^2+1}} = -1$$

The graph of

$$f(x) = \frac{x-2}{\sqrt{x^2+1}}$$



Horizontal Asymptotes

Definition

The line $y = b$ is called a horizontal asymptote of the curve $y = f(x)$ if either

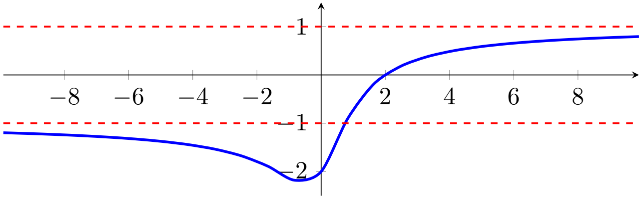
$$\lim_{x\to\infty} f(x) = b \quad \textit{or} \quad \lim_{x\to-\infty} f(x) = b$$

Example: $f(x) = \dfrac{x-2}{\sqrt{x^2+1}}$

We computed

- $\lim\limits_{x\to\infty} \frac{x-2}{\sqrt{x^2+1}} = 1$
- $\lim\limits_{x\to-\infty} \frac{x-2}{\sqrt{x^2+1}} = -1$

Therefore $y = 1$ and $y = -1$ are horizontal asymptotes.

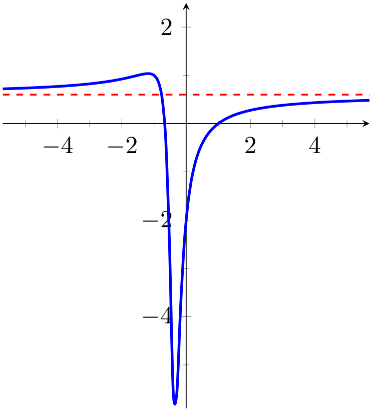


Horizontal Asymptotes

Find horizontal asymptotes for $f(x) = \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$

Solution: $\displaystyle\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{3 - \overset{0}{\cancel{\frac{1}{x}}} - \overset{0}{\cancel{\frac{2}{x^2}}}}{5 + \overset{0}{\cancel{\frac{4}{x}}} + \overset{0}{\cancel{\frac{1}{x^2}}}} = \boxed{\frac{3}{5}}$

$$\lim_{x\to-\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to-\infty} \frac{3 - \overset{0}{\cancel{\frac{1}{x}}} - \overset{0}{\cancel{\frac{2}{x^2}}}}{5 + \overset{0}{\cancel{\frac{4}{x}}} + \overset{0}{\cancel{\frac{1}{x^2}}}} = \boxed{\frac{3}{5}}$$



Therefore, $y = \frac{3}{5}$ is a horizontal asymptote.

Horizontal Asymptotes

Find horizontal asymptotes for $f(x) = \dfrac{\sin x}{x}$

Solution:

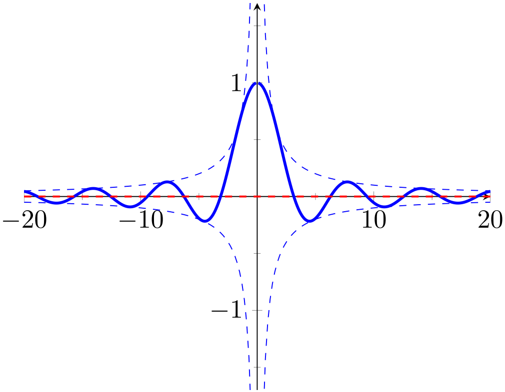
$$-1 \le \sin x \le 1$$

$$-\frac{1}{x} \le \quad \frac{1}{x}\sin x \quad \le \frac{1}{x}$$

$$0$$

By Squeeze Theorem $\lim\limits_{x\to\infty} \frac{\sin x}{x} = 0$

Therefore, $y = 0$ is a horizontal asymptote.



Vertical Asymptotes

Definition

The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if either

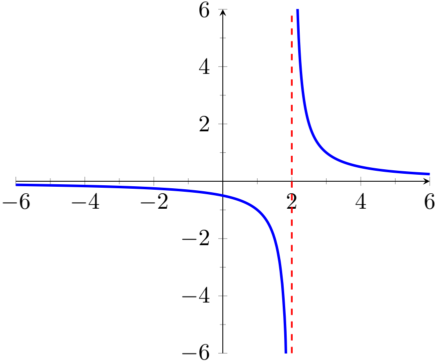
$$\lim_{x \to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \pm\infty$$

Example: $f(x) = \dfrac{1}{x-2}$

We compute

- $\lim\limits_{x \to 2^-} \frac{1}{x-2} = -\infty$
- $\lim\limits_{x \to 2^+} \frac{1}{x+1} = \infty$

Thus, $x = 2$ is a vertical asymptote.



Vertical Asymptotes of $\tan x$

Find the vertical asymptotes of $f(x) = \tan x$

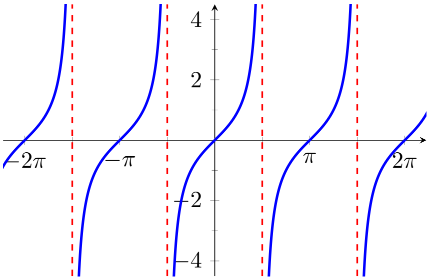
Since $\tan x = \frac{\sin x}{\cos x}$ and $\cos \frac{\pi}{2} = 0$ we compute

- $\lim\limits_{x\to\frac{\pi}{2}^-} \tan x = \lim\limits_{x\to\frac{\pi}{2}^-} \frac{\overset{\to 1}{\sin x}}{\underset{\to 0+}{\cos x}} = \infty$
- $\lim\limits_{x\to\frac{\pi}{2}^+} \tan x = \lim\limits_{x\to\frac{\pi}{2}^+} \frac{\overset{\to 1}{\sin x}}{\underset{\to 0-}{\cos x}} = -\infty$

Therefore, $x = \frac{\pi}{2}$ is a vertical asymptote.

Since tangent has period π, we obtain that $x = \frac{3\pi}{2}, x = \frac{5\pi}{2}, \ldots$ and $x = -\frac{\pi}{2}, x = -\frac{3\pi}{2}, \ldots$ are also vertical asymptotes of $\tan x$.

Thus, $x = \frac{\pi}{2} + \pi n$ (n is an integer) are the vertical asymptotes of $\tan x$.



Find the horizontal and vertical asymptotes of $f(x) = \frac{2x^2+x-1}{x^2+x-2}$

To find **horizontal asymptotes** we compute

$$\lim_{x\to\infty} \frac{2x^2+x-1}{x^2+x-2} = \lim_{x\to\infty} \frac{2+\overset{0}{\cancel{\frac{1}{x}}}-\overset{0}{\cancel{\frac{1}{x^2}}}}{1+\overset{0}{\cancel{\frac{1}{x}}}-\overset{0}{\cancel{\frac{2}{x^2}}}} = 2$$

Also, $\lim_{x\to-\infty} \frac{2x^2+x-1}{x^2+x-2} = \lim_{x\to-\infty} \frac{2+\overset{0}{\cancel{\frac{1}{x}}}-\overset{0}{\cancel{\frac{1}{x^2}}}}{1+\overset{0}{\cancel{\frac{1}{x}}}-\overset{0}{\cancel{\frac{2}{x^2}}}} = 2$

Thus, $y = 2$ is a horizontal asymptote.

Find the horizontal and vertical asymptotes of $f(x) = \frac{2x^2+x-1}{x^2+x-2}$

To find **vertical asymptotes** we find zeros of the denominator:

$x^2+x-2=0 \rightsquigarrow x=-2$ and $x=1$. Then $x^2+x-2=(x+2)(x-1)$

$$\lim_{x\to-2^-} f(x) = \lim_{x\to-2^-} \frac{\overset{\to 5}{2x^2+x-1}}{\underset{\to 0-}{(x+2)}\,\underset{\to -3}{(x-1)}} = \infty \qquad \lim_{x\to-2^-} f(x) = \lim_{x\to-2^+} \frac{\overset{\to 5}{2x^2+x-1}}{\underset{\to 0+}{(x+2)}\,\underset{\to -3}{(x-1)}} = -\infty$$

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} \frac{\overset{\to 2}{2x^2+x-1}}{\underset{\to 3}{(x+2)}\,\underset{\to 0-}{(x-1)}} = -\infty \qquad \lim_{x\to1^+} f(x) = \lim_{x\to1^+} \frac{\overset{\to 2}{2x^2+x-1}}{\underset{\to 3}{(x+2)}\,\underset{\to 0+}{(x-1)}} = \infty$$

Thus, $x=-2$ and $x=1$ are vertical asymptotes.

Find the horizontal and vertical asymptotes of $f(x) = \frac{2x^2+x-1}{x^2+x-2}$

$y = 2$ is a horizontal asymptote at ∞ and $-\infty$.

$x = -2, \ x = 1$ are the vertical asymptotes

- $\lim\limits_{x \to -2^-} f(x) = \infty$
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THE END