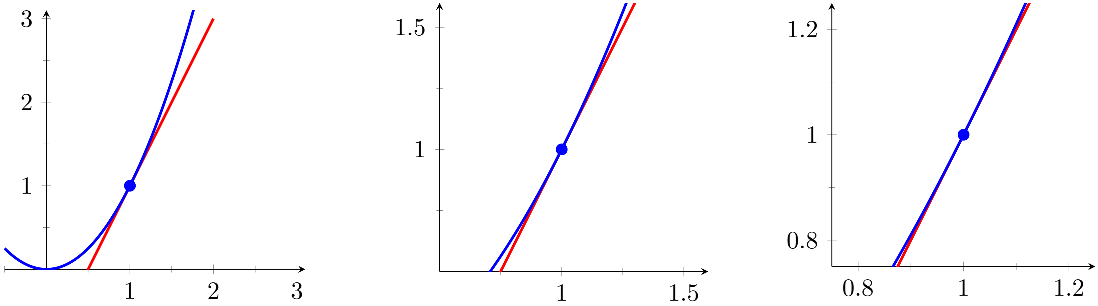
Linear Approximation and Differentials

Tamara Kucherenko

Linear Approximation

Consider the parabola $y = x^2$ and its tangent line $y = 2x - 1$ at the point (1,1)



By zooming in toward a point on the graph, we see that the graph looks more and more like its tangent line.

Linear Approximation

By definition, $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$. So when x is close to a,

$$\begin{aligned} f'(a) &\approx \frac{f(x) - f(a)}{x - a} \\ f'(a)(x-a) &\approx f(x) - f(a) \\ f(a) + f'(a)(x-a) &\approx f(x) \end{aligned}$$

The approximation

$$\boxed{f(x) \approx f(a) + f'(a)(x - a)}$$

is called the linear approximation or tangent line approximation of f at a.

The linear function $L(x) = f(a) + f'(a)(x - a)$ is called the linearization of f at a.

Example 1

Find the linearization of the function $f(x) = \sqrt{x-1}$ at $a = 10$ and use it to approximate $\sqrt{8.95}$.

The linearization of f at 10 is $L(x) = f(10) + f'(10)(x-10)$. We compute

- $f(10) = \sqrt{10-1} = \sqrt{9} = 3$
- $f'(x) = (\sqrt{x-1})' = \dfrac{1}{2\sqrt{x-1}} \quad \Rightarrow f'(10) = \dfrac{1}{2\sqrt{10-1}} = \dfrac{1}{6}$

We substitute the values into the equation for $L(x)$ and get

$$\boxed{L(x) = 3 + \frac{1}{6}(x-10) = \frac{1}{6}x + \frac{4}{3}}$$

To approximate $\sqrt{8.95}$ we set $x - 1 = 8.95$ and get $x = 9.95$. Therefore

$$\sqrt{8.95} = f(9.95) \approx L(9.95) = 3 + \tfrac{1}{6}(9.95 - 10) = 3 - 0.05 = \boxed{2.95}$$

Example 2

Use a linear approximation to estimate $\sqrt[3]{8.5}$

Consider $f(x) = \sqrt[3]{x}$ and $a = 8$. Then $\sqrt[3]{8.5} \approx f(8) + f'(8)(8.5 - 8)$. We compute

- $f(8) = \sqrt[3]{8} = 2$
- $f'(x) = (\sqrt[3]{x})' = \dfrac{1}{3}x^{-2/3} \quad \Rightarrow \quad f'(8) = \dfrac{1}{3}8^{-2/3} = \dfrac{1}{3} \cdot \dfrac{1}{(\sqrt[3]{8})} = \dfrac{1}{12}$

We substitute the values into the approximation and get

$$\sqrt[3]{8.5} \approx f(8) + f'(8)(8.5 - 8) = 2 + \tfrac{1}{12} \cdot 0.5 = 2 + \tfrac{1}{24} = \boxed{\dfrac{29}{24}}$$

Differentials

The ideas of linear approximation can be expressed in terms of differentials.

Definition

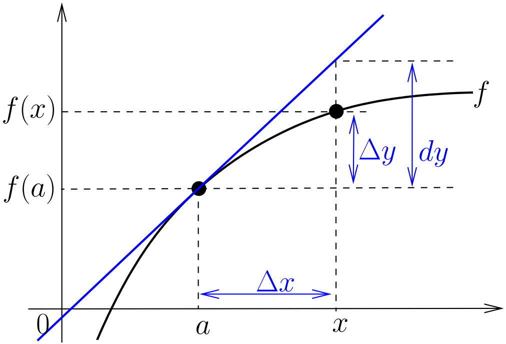
- If x is an independent variable then the differential dx represents the change in the value of x (usually infinitely small)
- If $y = f(x)$ then the differential $dy = f'(x)dx$.

Example. Find the differential dy of each function

- (a) $y = \sin^2 x \quad \Rightarrow \quad dy = (\sin^2 x)'dx = 2\sin x \cos x dx$
- (b) $y = \tan(2+5t) \quad \Rightarrow \quad dy = (\tan(2+5t))'dt = 5\sec^2(2+5t)dt$
- (c) $y = u\sqrt{u+1} \quad \Rightarrow \quad dy = (u\sqrt{u+1})'du = \left(\sqrt{u+1} + \frac{u}{2\sqrt{u+1}}\right)du$

Differentials

We use the tangent line at $(a, f(a))$ to approximate the curve $y = f(x)$ for x near a.



The change in x is $\Delta x = x - a$.
The change in y is $\Delta y = f(x) - f(a)$.

The linear approximation of f at a

$$f(x) \approx f(a) + f'(a)(x - a)$$

$$f(x) - f(a) \approx f'(a)(x - a)$$

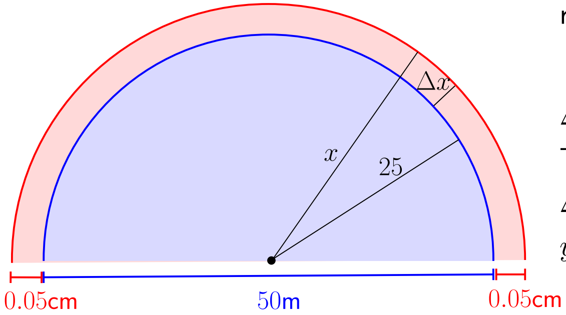
$$\Delta y \approx f'(a)\Delta x$$

When $dx = \Delta x$, $\quad \Delta y \approx f'(a)dx = dy$

$\Delta y \approx dy$ when $dx = \Delta x$

Example

Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.



Denote by y the volume of the hemisphere of radius x.

$$y = \frac{2}{3}\pi x^3$$

$\Delta x = 0.05\text{cm} = 0.0005\text{m}$

The amount of paint is $y(25.0005) - y(25) = \Delta y$

$\Delta y \approx dy = y'(25)dx = y'(25) \cdot 0.0005$

$y' = \frac{2}{3}\pi \cdot 3x^2 \quad \Rightarrow \quad y'(25) = 2\pi(25)^2 = 1250\pi$

$$\Delta y \approx 1250\pi \cdot 0.0005 = \boxed{0.625\pi \text{m}^3}$$

Summary

- The Linear Approximation is the estimate $\boxed{f(x) \approx f(a) + f'(a)(x-a)}$ when x is close to a.
- The function $L(x) = f(a) + f'(a)(x-a)$ is called the linearization of f at a.
- If $y = f(x)$ the differential of y is

$$dy = f'(x)dx$$

- In terms of differentials the Linear Approximation is the statement $\boxed{\Delta y \approx dy}$ where Δy is the change in $f(x)$ for a given change dx in x.

THE END