The Mean Value Theorem

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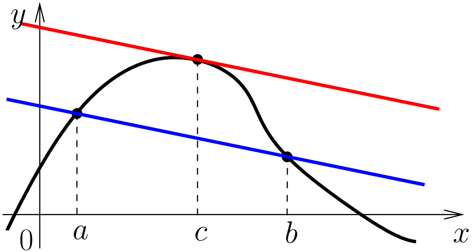
Assume that f is continuous on the closed interval $[a, b]$ and differentiable on (a, b).

Then there exists at least one value c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.



$\dfrac{f(b) - f(a)}{b - a}$ is the slope of the secant line.

$f'(c)$ is the slope of the tangent line.

MVT: There is at least one tangent line which is parallel to the secant line on (a, b).

Rolle's Theorem is a special case of MVT where $f(a) = f(b)$. In this case $f'(c) = 0$.

Example 1

Find all the numbers c that satisfy the conclusion of the Mean Value Theorem for $f(x) = x^3 - 2x$ on the interval $[-2, 2]$.

Since f is a polynomial, it is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$.

By the MVT there is a number c in $(-2, 2)$ such that $\boxed{f'(c) = \dfrac{f(2) - f(-2)}{2 - (-2)}}$.

Now $f(2) = 2^3 - 2 \cdot 2 = 4$, $f(-2) = (-2)^3 - 2 \cdot (-2) = -4$, and $f'(x) = 3x^2 - 2$.

So the equation becomes $\boxed{3c^2 - 2 = \dfrac{4 - (-4)}{2 - (-2)}} = \dfrac{8}{4} = 2$

Therefore, $3c^2 - 2 = 2 \Rightarrow c^2 = \frac{4}{3} \Rightarrow c = \pm\frac{2}{\sqrt{3}}$.

Since both numbers $\frac{2}{\sqrt{3}}$ and $-\frac{2}{\sqrt{3}}$ are in $(-2, 2)$, we obtain $\boxed{c = \pm\dfrac{2}{\sqrt{3}}}$

Example 2

Suppose that $f(0) = 7$ and $f'(x) \le 4$ for all values of x. How large can $f(6)$ possibly be?

We are given that f is differentiable (and therefore continuous) everywhere.

We apply the Mean Value Theorem on $[0, 6]$.

There is a number c in the interval $(0, 6)$ such that $\boxed{\dfrac{f(6) - f(0)}{6 - 0} = f'(c)}$

We are given that $f'(x) \le 4$ for all x, so in particular $f'(c) \le 4$.

Since $f(0) = -7$, the equation gives $\boxed{\dfrac{f(6) - (-7)}{6} \le 4} \Leftrightarrow f(6) + 7 \le 24 \Rightarrow f(6) \le 17$.

The largest possible value of $f(6)$ is $\boxed{17}$.

Example 3

Does there exist a function f such that $f(5) = 1$, $f(7) = 4$, and $f'(x) \leq 2$ for all x?

Since the function f is differentiable everywhere, it has to satisfy the conclusion of the MVT on the interval $[5, 7]$. There must be a point c in $(5, 7)$ such that

$$\frac{f(7) - f(5)}{7 - 5} = f'(c)$$

We are given that $f'(x) \leq 2$ for all x, so in particular $f'(c) \leq 2$.

Since $f(5) = 1$, $f(7) = 4$, the equation gives $\frac{4 - (-1)}{7 - 5} \leq 2 \Rightarrow \frac{5}{2} \leq 2$ Contradiction!

Therefore, the answer is $\boxed{No}$, there is no function such that $f(5) = 1$, $f(7) = 4$, and $f'(x) \leq 2$ for all x.

Corollary

If $f'(x) = 0$ for all x in an interval (a, b) then f is constant on (a, b).

Let x_1 and x_2 be any two numbers in (a, b) with $x_1 < x_2$.
By applying the MVT to f on the interval $[x_1, x_2]$ we get a number c in (x_1, x_2) such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

Since $f'(x) = 0$ for all x in (a, b), we have $f'(c) = 0$ and equation becomes

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0 \quad \Rightarrow \quad f(x_2) - f(x_1) = 0 \quad \Rightarrow \quad f(x_2) = f(x_1)$$

Therefore f has the same value at any two numbers in (a, b). This means that f is constant on (a, b).

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