Maximum and Minimum Values

Tamara Kucherenko

Lesson Plan

- Absolute and local maxima and minima GO
- Finding local maxima and minima using the derivative GO
- The closed interval method GO

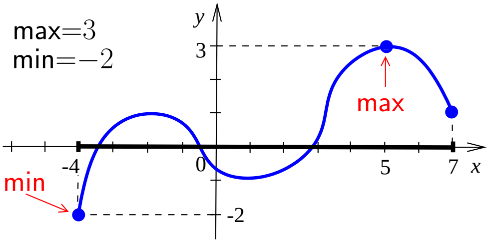
Absolute and Local Extrema

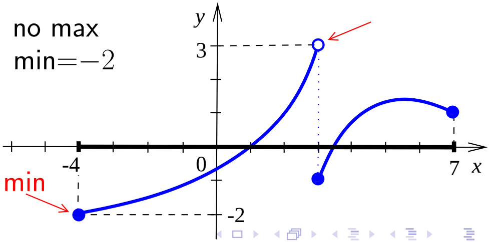
Definition

Let $f(x)$ be a function on an interval I and let c be a point in I. We say that

- *$f(c)$ is the absolute maximum of $f(x)$ on I if $f(c) \geq f(x)$ for all x in I.*
- *$f(c)$ is the absolute minimum of $f(x)$ on I if $f(c) \leq f(x)$ for all x in I.*

We refer to the maximum and minimum values as extreme values or extrema.



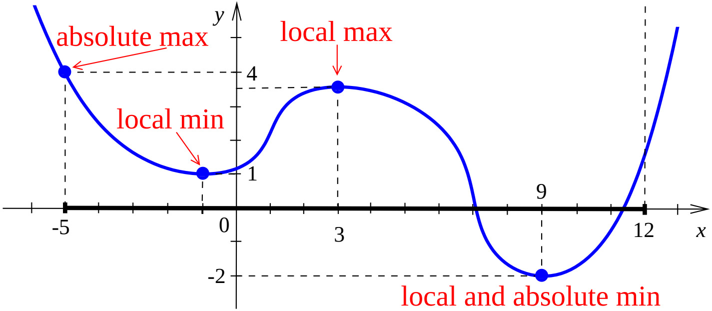


Absolute and Local Extrema

Definition

We say that $f(x)$ has a

- *<u>local maximum</u> at $x = c$ if $f(c) \geq f(x)$ for all x in an open interval containing c.*
- *<u>local minimum</u> at $x = c$ if $f(c) \leq f(x)$ for all x in an open interval containing c.*



On [-5,12]:

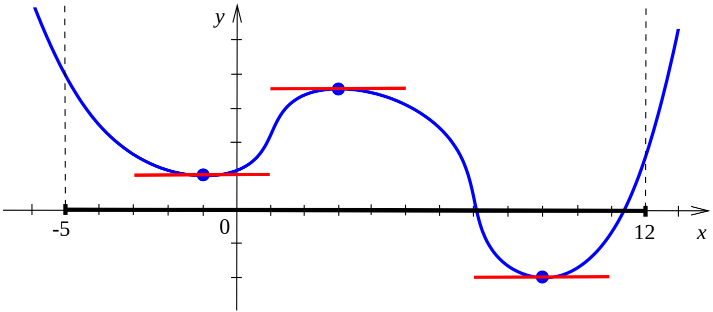
Absolute maximum is 4 at x=-5

Absolute minimum is -2 at x=9

Local maximum is (3,3.5)

Local minima are (-1,1) and (9,-2)

Finding Local Extrema Using the Derivative

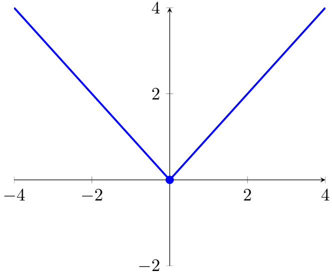


Fermat's Theorem

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Finding Local Extrema Using the Derivative

If $f(x) = |x|$, then $f(0) = 0$ is a minimum value, but $f'(0)$ does not exist.



Definition

A number c in the domain of f is called a critical point if either $f'(c) = 0$ or $f'(c)$ does not exist.

Example 1

Find the critical numbers of $f(x) = x^4 - x^3 - x^2 - 1$.

$$f'(x) = 4x^3 - 3x^2 - 2x = 0$$

$$x(4x^2 - 3x - 2) = 0 \quad \Longrightarrow \quad x = 0 \text{ or } x = \frac{3 \pm \sqrt{9 + 4 \cdot 4 \cdot 2}}{8} = \frac{3 \pm \sqrt{41}}{8}$$

There are three critical points: $\boxed{x = 0, \ x = \frac{3 \pm \sqrt{41}}{8}}$

Example 2

Find the critical numbers of $f(x) = x^{\frac{3}{5}}(4 - x)$.

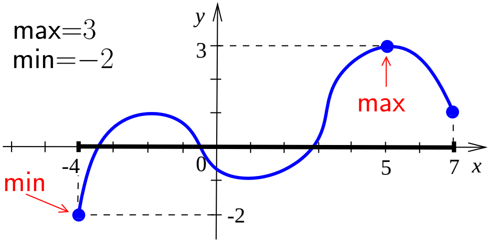
$$f(x) = 4x^{\frac{3}{5}} - x^{\frac{8}{5}}, \quad f'(x) = \frac{12}{5}x^{-\frac{2}{5}} - \frac{8}{5}x^{\frac{3}{5}} = \frac{12}{5x^{\frac{2}{5}}} - \frac{8}{5}x^{\frac{3}{5}} = \frac{12 - 8x}{5x^{\frac{2}{5}}}$$

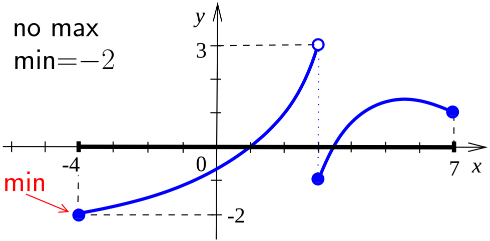
f' does not exist when $x = 0$.

$f'(x) = 0$ if $12x - 8 = 0$, that is $x = \frac{3}{2}$

There are two critical points: $\boxed{x = 0,\ x = \frac{3}{2}}$

The Closed Interval Method





function is continuous on [-4,7]

function is not continuous on [-4,7]

The Extreme Value Theorem

If f is continuous on a closed interval $[a,b]$, then f attains its absolute maximum value and its absolute minimum value at some numbers in $[a,b]$.

The Closed Interval Method

The Closed Interval Method

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. *Find the critical numbers of f in (a, b).*
2. *Compute the values of f at all the critical points in (a, b) and the endpoints.*
3. *The largest of these values is the absolute maximum, the smallest is the absolute minimum.*

Example 1

Find the absolute maximum and minimum values of the function $f(x) = 2x^3 - 15x^2 + 24x + 7$ on $[0, 6]$.

1. Find the critical numbers: $f'(x) = 6x^2 - 30x + 24 = 6(x-1)(x-4) = 0$
   The critical numbers are $x = 1$ and $x = 4$.
2. Compute the values of f at the critical points in $(0, 6)$ and the endpoints.

   We compute $f(x)$ at $x = 1$, $x = 4$, $x = 0$, $x = 6$.
   $f(1) = 2(1)^3 - 15(1)^2 + 24(1) + 7 = 18$
   $f(4) = 2(4)^3 - 15(4)^2 + 24(4) + 7 = -9$
   $f(0) = 2(0)^3 - 15(0)^2 + 24(0) + 7 = 7$
   $f(6) = 2(6)^3 - 15(6)^2 + 24(6) + 7 = 43$
3. Compare: The largest of these values is 43, so the $\boxed{\text{absolute maximum is } 43}$, the smallest is -9, so the $\boxed{\text{absolute minimum is } -9}$.

Example 2

Find the absolute maximum and minimum values of the function $f(x) = 1 - (x-1)^{\frac{2}{3}}$ on $[-1, 2]$.

1. Find the critical numbers: $f'(x) = -\frac{2}{3}(x-1)^{-\frac{1}{3}} = -\frac{2}{3(x-1)^{\frac{1}{3}}}$

   $f'(x)$ is never zero, but $f'(1)$ DNE. The critical number is $x = 1$.
2. Compute the values of f at the critical points in $(-1, 2)$ and the endpoints.

   We compute $f(x)$ at $x = 1$, $x = -1$, $x = 2$.

   $f(1) = 1 - (1-1)^{\frac{2}{3}} = 1$

   $f(-1) = 1 - (-1-1)^{\frac{2}{3}} = 1 - (-2)^{\frac{2}{3}} = 1 - \sqrt[3]{(-2)^2} = 1 - \sqrt[3]{4}$

   $f(2) = 1 - (2-1)^{\frac{2}{3}} = 0$
3. Compare: The largest of these values is 1, so the $\boxed{\text{absolute maximum is } 1}$, the smallest is $1 - \sqrt[3]{4}$, so the $\boxed{\text{absolute minimum is } 1 - \sqrt[3]{4}}$.

Example 3

Find the absolute maximum and minimum values of the function $f(x) = \dfrac{x}{x^2 - x + 1}$ on $[0, 3]$.

1. Find the critical numbers: $f'(x) = \dfrac{x^2 - x + 1 - x(2x-1)}{(x^2 - x + 1)^2} = \dfrac{1 - x^2}{(x^2 - x + 1)^2}$
   $f'(x) = 0$ when $1 - x^2 = 0$ or $x = \pm 1$.
   Since $x^2 - x + 1 \neq 0$, the critical numbers are $x = 1$ and $x = -1$.
2. Compute the values of f at the critical points in $(0, 3)$ and the endpoints.
   We compute $f(x)$ at $x = 1$, $x = 0$, $x = 3$.
   $f(1) = \dfrac{1}{1^2 - 1 + 1} = 1$, $\quad f(0) = \dfrac{0}{0^2 - 0 + 1} = 0$, $\quad f(3) = \dfrac{3}{3^2 - 3 + 1} = \dfrac{3}{7}$
3. Compare: The largest of these values is 1, so the **absolute maximum is 1**, the smallest is 0, so the **absolute minimum is 0**.

THE END