Optimization

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Optimization

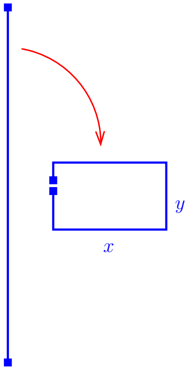
An optimization problem consists of finding the "best available" value of some function on a given domain.

Strategy:

- Choose variables. Determine which quantities are relevant by drawing a diagram and assign appropriate variables.
- Find the function and the interval. Restate the problem in terms of finding the maximum or the minimum of a function f on some interval. If f depends on more than one variable, use the given information to rewrite f as a function of just one variable.
- Optimize the function. Use the derivative to find the absolute maximum or minimum of the function. If the interval is closed, use the Closed Interval Method.

Example 1

A piece of wire of length 8 in is bent into the shape of a rectangle. Which dimensions produce the rectangle of maximum area?



Notation: x-length, y-width, A-area.

Given: perimeter $2x + 2y = 8$. Goal: Maximize the area.

Area of the rectangle: $A = x \cdot y$

Rewrite the area as a function of one variable:
$2x + 2y = 8 \quad \Rightarrow \quad y = 4 - x$. Thus, $A = x(4 - x) = 4x - x^2$

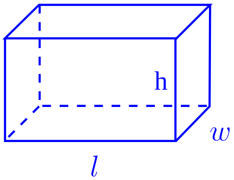
Goal: Maximize $A(x) = 4x - x^2$ for $0 \leq x \leq 4$.

We use the closed interval method: $A'(x) = 4 - 2x = 0 \Rightarrow x = 2$.
$A(0) = 4{\cdot}0 - (0)^2 = 0$, $A(2) = 4{\cdot}2 - (2)^2 = 4$, $A(4) = 4{\cdot}4 - (4)^2 = 0$

The rectangle of maximal area is the square of sides $x = y = 2$.

Example 2

A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$20 per square meter. Material for the sides costs \$9 per square meter. Find the cost of materials for the cheapest such container.



Notation: l-length, w-width, h-hight, V-volume

Given: $V = 10\text{m}^3$, $l = 2w$,
base cost=\$20 per m^2, side cost=\$9 per m^2

Goal: Minimize the cost

Cost= $20\cdot$(Area of the base)$+9\cdot$(Area of four sides)
$= 20 \cdot (wl) + 9(2lh + 2wh)$

Rewrite the cost as a function of one variable:

$l = 2w$ and $V = 10 = lwh \Rightarrow h = \dfrac{10}{lw} = \dfrac{10}{2w^2} = \dfrac{5}{w^2}$

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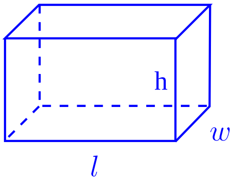

Notation: l-length, w-width, h-hight, V-volume

Goal: Minimize the cost

Rewrite the cost function using $l = 2w$ and $h = \frac{5}{w^2}$:

$$\begin{aligned}\text{Cost} &= 20\cdot(wl) + 9(2lh + 2wh)\\ &= 20\cdot(w\cdot 2w) + 9\left(2\cdot 2w\cdot\frac{5}{w^2} + 2w\cdot\frac{5}{w^2}\right)\\ &= 40w^2 + \frac{270}{w} = C(w)\end{aligned}$$

Minimize $C(w)$ for $w > 0$.

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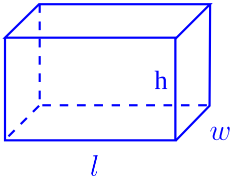
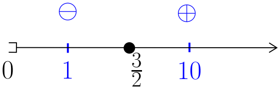


Notation: l-length, w-width, h-hight, V-volume

Goal: Minimize $C(w) = 40w^2 + \dfrac{270}{w}$ for $w > 0$.

Differentiate: $C'(w) = 80w - \dfrac{270}{w^2} = \dfrac{80w^3 - 270}{w^2}$

Critical numbers: $80w^3 - 270 = 0 \Rightarrow w^3 = \dfrac{270}{80} \Rightarrow w = \dfrac{3}{2}$



$C'(1) = -190 < 0$

$C'(10) = 797.3 > 0$

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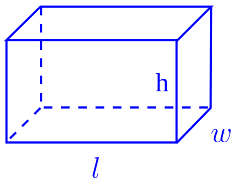

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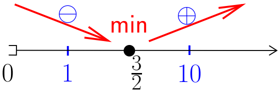
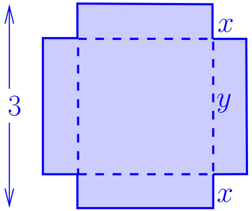


Minimal cost is

$C(\frac{3}{2}) = 40(\frac{3}{2})^2 + \frac{270}{\frac{3}{2}} = \boxed{\$270}$

Example 3

A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.



Given: $2x + y = 3$. Goal: Maximize the volume

Volume $V =$ (hight)$\cdot$(Area of the base)$= x \cdot y^2$

Rewrite the volume as a function of one variable:
$2x + y = 3 \Rightarrow y = 3 - 2x$. Thus, $V = x(3 - 2x)^2$

Goal: Maximize $V(x) = x(3 - 2x)^2$ for $0 \le x \le \frac{3}{2}$

x-length of the cut square,
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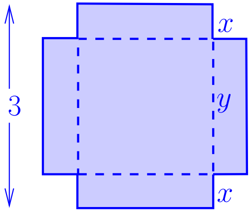

x-length of the cut square,
y-length of the base

Goal: Maximize $V(x) = x(3-2x)^2$ for $0 \le x \le \frac{3}{2}$

Use the closed interval method:

$V'(x) = (3-2x)^2 + 2x(3-2x)(-2) = (3-2x)(3-6x) = 0$

Critical numbers: $x = \frac{3}{2}$ and $x = \frac{1}{2}$.

Points to check: $x = 0$, $x = \frac{1}{2}$, $x = \frac{3}{2}$.

$V(0) = 0$, $V(\frac{1}{2}) = \frac{1}{2}(3 - 2 \cdot \frac{1}{2})^2 = 2$, $V(\frac{3}{2}) = 0$.

The maximal volume is $\boxed{2 \text{ ft}^3}$

THE END