Related Rates

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In related rates problems we compute the rate of change of one quantity in terms of the rate of change of the other.

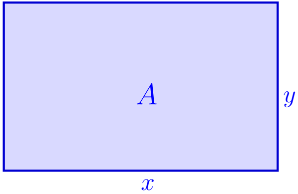
Rate of change $=$ derivative

Strategy:

- Draw a diagram and introduce the notation
- Write an equation relating the quantities of the problem
- Take derivative with respect to time (Use the Chain Rule!)
- Substitute the given information and solve for the unknown.

Example 1

The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?



Notation: t-time, x-length, y-width, A-area

Given: $\frac{dx}{dt} = 8, \quad \frac{dy}{dt} = 3$

Find: $\frac{dA}{dt}$ when $x = 20$ and $y = 10$.

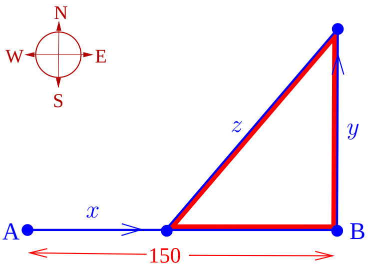
Area of the rectangle: $A = x \cdot y$

Differentiate with respect to t : $\dfrac{dA}{dt} = \dfrac{dx}{dt}\, y + x\, \dfrac{dy}{dt}$

Substitute the given information: $\dfrac{dA}{dt} = 8 \cdot 10 + 20 \cdot 3 = \boxed{120 \, \frac{\text{cm}^2}{\text{s}}}$

Example 2

At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 PM?



Notation: t - time elapsed since noon,
x - distance travelled by A since noon,
y - distance travelled by B since noon,
z - distance between A and B

Given: $\frac{dx}{dt} = 35$, $\frac{dy}{dt} = 25$. Find: $\frac{dz}{dt}$ when $t = 4$.

The Pythagorean theorem: $z^2 = (150 - x)^2 + y^2$

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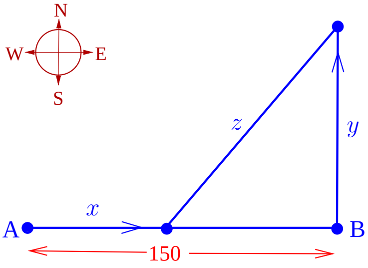


Given: $\frac{dx}{dt} = 35$, $\frac{dy}{dt} = 25$. Find: $\frac{dz}{dt}$ when $t = 4$.

The Pythagorean theorem: $z^2 = (150 - x)^2 + y^2$

Differentiate: $2z\frac{dz}{dt} = 2(150 - x)\left(-\frac{dx}{dt}\right) + 2y\frac{dy}{dt}$

At time $t = 4$: $x = 35 \cdot 4 = 140$ $y = 25 \cdot 4 = 100$

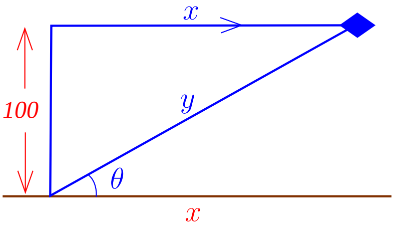
$z^2 = (150 - 140)^2 + 100^2 = 10100$, so $z = 10\sqrt{101}$.

Plug in: $2 \cdot 10\sqrt{101}\frac{dz}{dt} = 2(150 - 140)(-35) + 2 \cdot 100 \cdot 25$

Solve for $\frac{dz}{dt}$: $\frac{dz}{dt}(4) = \boxed{\frac{430}{\sqrt{101}}\text{km/h}}$

Example 3

A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string has been let out?



Notation: t - time elapsed,
x - distance travelled by the kite,
y - length of the string,
θ - angle between the string and the horizontal

Given: $\frac{dx}{dt} = 8$. Find: $\frac{d\theta}{dt}$ when $y = 200$.

$$\tan\theta = \frac{100}{x}$$

Differentiate: $$\sec^2\theta \cdot \frac{d\theta}{dt} = -\frac{100}{x^2} \cdot \frac{dx}{dt}$$

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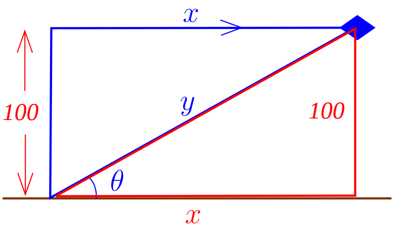


Given: $\frac{dx}{dt} = 8$, Find: $\frac{d\theta}{dt}$ when $y = 200$.

$\tan\theta = \frac{100}{x}$ Differentiate: $\sec^2\theta \cdot \frac{d\theta}{dt} = -\frac{100}{x^2} \cdot \frac{dx}{dt}$

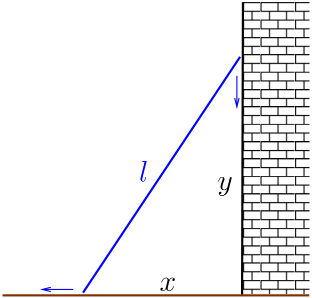
When $y = 200$: $x^2 = 200^2 - 100^2 = 30000$

$$\sec\theta = \frac{y}{x} \implies \sec^2\theta = \frac{y^2}{x^2} = \frac{40000}{30000} = \frac{4}{3}$$

Plug in: $\frac{4}{3} \cdot \frac{d\theta}{dt} = -\frac{100}{30000} \cdot 8 \implies \frac{d\theta}{dt} = \boxed{-0.02\,\text{rad/s}}$

Example 4

The top of a ladder slides down a vertical wall at a rate of 0.15 m/s. At the moment when the bottom of the ladder is 3 m from the wall, it slides away from the wall at a rate of 0.2 m/s. How long is the ladder?



Notation: t - time elapsed,
x - distance of the bottom to the wall,
y - distance of the top to the ground,
l - length of the ladder

Given: $\frac{dy}{dt} = -0.15$ and $\frac{dx}{dt} = 0.2$ when $x = 3$. Find: l.

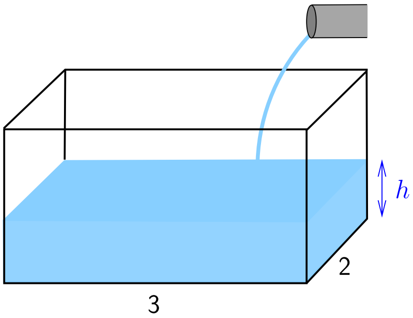
$l^2 = x^2 + y^2$ Differentiate: $0 = 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt}$

Plug in: $0 = 2 \cdot 3 \cdot 0.2 + 2y(-0.15) \implies y = 4$

$l^2 = 3^2 + 4^2 = 25 \implies l = \boxed{5\text{ m}}$

Example 5

Water pours into a fish tank at a rate of 3 ft^3/min. How fast is the water level rising if the base of the tank is a rectangle of dimensions 2×3 ft?



Notation: t - time elapsed,
h - hight of the water,
V - volume of the water

Given: $\frac{dV}{dt} = 3$. Find: $\frac{dh}{dt}$.

$V = 3 \cdot 2 \cdot h = 6h$ Differentiate: $\dfrac{dV}{dt} = 6\,\dfrac{dh}{dt}$

Plug in: $3 = 6 \cdot \dfrac{dh}{dt} \implies \dfrac{dh}{dt} = \boxed{0.5\,\text{ft}/\text{min}}$

THE END