The Rules of Differentiation

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Lesson Plan

The Power Rule

Derivative of a constant function $f(x) = c$:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

$$\boxed{(c)' = 0}$$

Derivative of $f(x) = x$:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(\cancel{x}+h)-\cancel{x}}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

$$\boxed{(x)' = 1}$$

The Power Rule

Derivative of $f(x) = x^2$:

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} = \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = \lim_{h\to 0} (2x + \overset{0}{\cancel{h}}) = 2x$$

$$\boxed{(x^2)' = 2x}$$

Derivative of $f(x) = x^3$:

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h} = \lim_{h\to 0} \frac{\cancel{h}(3x^2 + \overset{0}{\cancel{3xh}} + \overset{0}{\cancel{h^2}})}{\cancel{h}} = 3x^2$$

$$\boxed{(x^3)' = 3x^2}$$

Similarly,

$$\boxed{(x^4)' = 4x^3}$$

The Power Rule

The Power Rule

For any real number n $(x^n)' = nx^{n-1}$

Example 1: $(x^{17})' = 17x^{17-1} = 17x^{16}$

Example 2: $\left(\frac{1}{x^2}\right)' = (x^{-2})' = -2x^{-2-1} = -2x^{-3} = -\frac{2}{x^3}$

Example 3: $(\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$$\boxed{(\sqrt{x})' = \frac{1}{2\sqrt{x}}}$$

Example 4: $(x^\pi)' = \pi x^{\pi-1}$

Example 5: $(\sqrt[3]{x^2})' = (x^{\frac{2}{3}})' = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$

The Linearity Rules

Linearity Rules

Assume that f and g are differentiable functions.

The Sum/Difference Rule: *The function $f \pm g$ is differentiable and*

$$(f \pm g)' = f' \pm g'$$

The Constant Multiple Rule: *For any constant c, cf is differentiable and*

$$(cf)' = cf'$$

Example: $(x^3 - 5x + 7)' = (x^3)' - 5(x)' + (7)' = 3x^2 - 5$

The Linearity Rules

Example 1: Calculate $f'(x)$ where $f(x) = \frac{1}{2x^2} + 3\sqrt[3]{x} - x^{-\frac{3}{2}}$

$$\begin{aligned}
f'(x) = \left(\frac{1}{2}x^{-2} + 3x^{\frac{1}{3}} - x^{\frac{3}{2}}\right)' &= \frac{1}{2}\left(x^{-2}\right)' + 3\left(x^{\frac{1}{3}}\right)' - \left(x^{\frac{3}{2}}\right)' \\
&= \frac{1}{2}\left(-2x^{-3}\right) + 3\left(\frac{1}{3}x^{-\frac{2}{3}}\right) - \left(\frac{3}{2}x^{\frac{1}{2}}\right) \\
&= -x^{-3} + x^{-\frac{2}{3}} - \frac{3}{2}x^{\frac{1}{2}} \\
&= \boxed{-\frac{1}{x^3} + \frac{1}{\sqrt[3]{x^2}} - \frac{3}{2}\sqrt{x}}
\end{aligned}$$

The Linearity Rules

Example 2: Calculate $f'(4)$ where $f(x) = \frac{2x^2+\sqrt{x}-1}{x}$

$$\begin{aligned} f(x) &= \frac{2x^2+\sqrt{x}-1}{x} \\ &= \frac{2x^2}{x} + \frac{\sqrt{x}}{x} - \frac{1}{x} \\ &= 2x + x^{\frac{1}{2}-1} - x^{-1} \\ &= 2x + x^{-\frac{1}{2}} - x^{-1} \end{aligned}$$

$$\begin{aligned} f'(x) &= \left(2x + x^{-\frac{1}{2}} - x^{-1}\right)' \\ &= 2(x)' + (x^{-\frac{1}{2}})' - (x^{-1})' \\ &= 2 - \frac{1}{2}x^{-\frac{3}{2}} + x^{-2} \end{aligned}$$

$$f'(4) = 2 - \frac{1}{2} \cdot 4^{-\frac{3}{2}} + 4^{-2} = 2 - \frac{1}{2} \cdot \frac{1}{8} + \frac{1}{16} = \boxed{2}$$

The Product Rule

Product Rule

If f and g are differentiable then fg is differentiable and

$$\boxed{(fg)'(x) = f'(x)g(x) + f(x)g'(x)}$$

The verbal form of the product rule $(fg)'(x) = (\text{first})' \cdot \text{second} + \text{first} \cdot (\text{second})'$

Example:

$$(\underbrace{2x^3}_{\text{first}}\underbrace{(4x-5)}_{\text{second}})' = \underbrace{(2x^3)'}_{\text{first}'}\underbrace{(4x-5)}_{\text{second}} + \underbrace{2x^3}_{\text{first}}\underbrace{(4x-5)'}_{\text{second}'} = \underbrace{6x^2}_{\text{first}'}\underbrace{(4x-5)}_{\text{second}} + \underbrace{2x^3}_{\text{first}}\underbrace{(4)}_{\text{second}'}$$

$$= \boxed{32x^3 - 30x^2}$$

The Product Rule

Remember that $(fg)' \neq f'g'$.

To see this take $f(x) = x$ and $g(x) = x^2$. Then $fg = x \cdot x^2 = x^3$ and

$$(fg)'(x) = \left(x^3\right)' = 3x^2 \quad \neq \quad f'(x)g'(x) = (x)'(x^2)' = 1(2x) = 2x$$

The Product Rule

Example 1:

$$(\underbrace{x^2}_{\text{first}}\ \underbrace{\sin x}_{\text{second}})' = \underbrace{(x^2)'}_{\text{first}'}\ \underbrace{\sin x}_{\text{second}} + \underbrace{x^2}_{\text{first}}\ \underbrace{(\sin x)'}_{\text{second}'} = \underbrace{2x}_{\text{first}'}\ \underbrace{\sin x}_{\text{second}} + \underbrace{x^2}_{\text{first}}\ \underbrace{\cos x}_{\text{second}'}$$

$$= \boxed{2x\sin x + x^2\cos x}$$

Example 2:

$$\left(\frac{\cos x}{x^3}\right)' = (x^{-3}\cos x)' = (x^{-3})'\cos x + x^{-3}(\cos x)' = (-3x^{-4})\cos x + x^{-3}(-\sin x)$$

$$= \boxed{-\frac{3\cos x}{x^4} - \frac{\sin x}{x^3}}$$

The Quotient Rule

Quotient Rule

If f and g are differentiable then f/g is differentiable for all x such that $g(x) \neq 0$ and

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

The verbal form of the quotient rule $\left(\frac{f}{g}\right)'(x) = \frac{(\text{top})' \cdot \text{bottom} - \text{top} \cdot (\text{bottom})'}{\text{bottom}^2}$

The Quotient Rule

Example 1:

$$\left(\frac{x^2}{x+1}\right)' = \frac{\overbrace{(x^2)'}^{\text{top}'}\overbrace{(x+1)}^{\text{bottom}} - \overbrace{x^2}^{\text{top}}\overbrace{(x+1)'}^{\text{bottom}'}}{\underbrace{(x+1)^2}_{\text{bottom}^2}} = \frac{2x(x+1)-(x^2)1}{(x+1)^2} = \boxed{\frac{x^2+2x}{(x+1)^2}}$$

Example 2:

$$\left(\frac{\sqrt{x}}{x^2+5}\right)' = \frac{\overbrace{(\sqrt{x})'}^{\text{top}'}\overbrace{(x^2+5)}^{\text{bottom}} - \overbrace{\sqrt{x}}^{\text{top}}\overbrace{(x^2+5)'}^{\text{bottom}'}}{\underbrace{(x^2+5)^2}_{\text{bottom}^2}} = \frac{\frac{1}{2\sqrt{x}}(x^2+5)-\sqrt{x}\,2x}{(x^2+5)^2} = \boxed{\frac{5-3x^2}{2\sqrt{x}(x^2+5)^2}}$$

Derivatives of Trigonometric functions

We know that $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$.

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\overbrace{(\sin x)'}^{\text{top}'}\overbrace{\cos x}^{\text{bottom}} - \overbrace{\sin x}^{\text{top}}\overbrace{(\cos x)'}^{\text{bottom}'}}{\underbrace{(\cos x)^2}_{\text{bottom}^2}}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \boxed{\sec^2 x}$$

Similarly, $\boxed{(\cot x)' = -\csc^2 x, \quad (\sec x)' = \sec x \tan x, \quad (\csc x)' = -\csc x \cot x}$

Summary

Rules of Differentiation

Sum/Difference Rule: $(f \pm g)' = f' \pm g'$

Constant Multiple Rule: $(cf)' = cf'$

Product Rule: $(fg)' = f'g + fg'$

Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Derivatives of Basic Functions

$(c)' = 0$ (where c is a constant)

$(x^n)' = nx^{n-1}$ (the power rule)

$(\sin x)' = \cos x \qquad (\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x \qquad (\cot x)' = -\csc x$

$(\sec x)' = \sec x \tan x$

$(\csc x)' = -\csc x \cot x$

THE END