Substitution Method

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Substitution Method

The Substitution Method is the Chain Rule "in reverse".

$$\frac{d}{dx}\sin(x^2) = 2\,x\,\cos(x^2) \quad \Longrightarrow \quad \int \underbrace{2\,x}_{\text{Derivative of inside function}} \cos \underbrace{(x^2)}_{\text{Inside function}}\, dx = \sin(x^2) + C$$

The method works when we can rewrite the integral in the form $\int f(u(x))\,u'(x)\,dx$.

If $F(u)$ is an antiderivative of $f(u)$, then by the Chain Rule

$\frac{d}{dx}F(u(x)) = F'(u(x))u'(x) = f(u(x))\,u'(x)$. Therefore,

$$\int \underbrace{f(u(x))}_{f(u)}\,\underbrace{u'(x)\,dx}_{du} = F(u(x)) + C = \int f(u)\,du$$

Substitution Method

The Substitution Rule

$$\int f(u(x))\, u'(x)\, dx = \int f(u)\, du$$

Example:

$$\int \underbrace{(1+3x^2)}_{\text{Derivative of inside function}} \sqrt{\underbrace{x+x^3}_{\text{Inside function}}}\, dx = \int \sqrt{\underbrace{x+x^3}_{u}}\; \underbrace{(1+3x^2)\, dx}_{du} = \int \sqrt{u}\, du = \frac{2}{3} u^{\frac{3}{2}} + C$$

$$= \boxed{\frac{2}{3}(x+x^3)^{\frac{3}{2}} + C}$$

Check: $\left(\frac{2}{3}(x+x^3)^{\frac{3}{2}}\right)' = \frac{2}{3} \cdot \frac{3}{2}(x+x^3)^{\frac{1}{2}}(1+3x^2) = \sqrt{x+x^3}(1+3x^2)$

Substitution Method

The Substitution Rule

$$\int f(u(x))\, u'(x)\, dx = \int f(u)\, du$$

Strategy:

- Choose the function u and compute du.
- Rewrite the integral in terms of u and du, and evaluate.
- Express the final answer in terms of x.

Example 1

Evaluate $\int 2x(x^2+5)^4\,dx$.

$$\int 2x\underbrace{(x^2+5)}_{\text{Inside function}}{}^4\,dx = \left|\begin{array}{l} u = x^2+5 \\ du = 2x\,dx \end{array}\right| = \int u^4\,du = \frac{u^5}{5} + C = \boxed{\frac{1}{5}(x^2+5)^5 + C}$$

Example 2

Evaluate $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx.$

$$\int \frac{x}{\underbrace{\sqrt{1-4x^2}}_{\text{Inside function}}}\,dx = \left|\begin{array}{c} u = 1-4x^2 \\ du = -8x\,dx \\ -\frac{1}{8}du = x\,dx \end{array}\right| = -\frac{1}{8}\int \frac{1}{\sqrt{u}}\,du = -\frac{1}{8}\int u^{-\frac{1}{2}}\,du$$

$$= -\frac{1}{8}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = -\frac{1}{4}\sqrt{u} + C$$

$$= \boxed{-\frac{1}{4}(1-4x^2)^{\frac{1}{2}} + C}$$

Example 3

Evaluate $\int x\sqrt{2x+7}\,dx$.

$$\int x\underbrace{\sqrt{2x+7}}_{\text{Inside function}}\,dx = \left| \begin{array}{l} u = 2x+7 \\ du = 2\,dx \Rightarrow \frac{1}{2}du = dx \\ u = 2x+7 \Rightarrow x = \frac{1}{2}(u-7) \end{array} \right| = \int \frac{1}{2}(u-7)\sqrt{u}\,\frac{1}{2}\,du$$

$$= \frac{1}{4}\int (u-7)u^{\frac{1}{2}}\,du = \frac{1}{4}\int (u^{\frac{3}{2}} - 7u^{\frac{1}{2}})\,du = \frac{1}{4}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}} - 7\frac{u^{-\frac{1}{2}}}{-\frac{1}{2}}\right) + C$$

$$= \frac{1}{2}u^{\frac{1}{2}} + \frac{7}{2}u^{-\frac{1}{2}} + C = \boxed{\frac{1}{2}(2x+7)^{\frac{1}{2}} + \frac{7}{2}(2x+7)^{-\frac{1}{2}} + C}$$

Substitution Method for Definite Integrals

Example: Evaluate $\displaystyle\int_0^4 (2x+1)^7\,dx$

$$\int (2x+1)^7 dx = \left| \begin{array}{l} u = 2x+1 \\ du = 2dx \Rightarrow \frac{1}{2}du = dx \end{array} \right| = \frac{1}{2}\int u^7 du = \frac{u^8}{16} + C = \frac{1}{16}(2x+1)^8 + C$$

By the Fundamental Theorem of Calculus $\displaystyle\int_0^4 (2x+1)^7\,dx = \frac{1}{16}(2x+1)^8\Big|_0^4 = \frac{9^8}{16} - \frac{1}{16}$

The Substitution Rule for Definite Integrals

$$\int_a^b f(u(x))\,u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

$$\int_0^4 (2x+1)^7 dx = \left| \begin{array}{l} u = 2x+1 \\ du = 2dx \Rightarrow \frac{1}{2}du = dx \\ u(0) = 1,\ u(4) = 9 \end{array} \right| = \frac{1}{2}\int_1^9 u^7 du = \frac{u^8}{16}\Big|_1^9 = \frac{9^8}{16} - \frac{1}{16}$$

Substitution Method for Definite Integrals

Evaluate $\int_0^{\frac{\sqrt{\pi}}{2}} x\cos(x^2)\,dx$.

$$\int_0^{\frac{\sqrt{\pi}}{2}} x\cos(x^2)\,dx = \left| \begin{array}{l} u = x^2 \\ du = 2x\,dx \Rightarrow \frac{1}{2}du = xdx \\ u(0) = 0,\ u(\frac{\sqrt{\pi}}{2}) = \frac{\pi}{4} \end{array} \right| = \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos u\,du$$

$$= \frac{1}{2}\sin u\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}(\sin\frac{\pi}{4} - \sin 0) = \boxed{\frac{\sqrt{2}}{4}}$$

THE END