REALIZATION OF ANOSOV DIFFEOMORPHISMS ON THE TORUS

TAMARA KUCHERENKO AND ANTHONY QUAS

Abstract. We study area preserving Anosov maps on the two-dimensional torus within a fixed homotopy class. We show that the set of pressure functions for Anosov diffeomorphisms with respect to the geometric potential is equal to the set of pressure functions for the linear Anosov automorphism with respect to Hölder potentials. We use this result to provide a negative answer to the $C^{1+\alpha}$ version of the question posed by Rodriguez Hertz on whether two homotopic area preserving C^∞ Anosov difeomorphisms whose geometric potentials have identical pressure functions must be C^∞ conjugate.

1. Introduction

We consider an Anosov diffeomorphism T of the two-dimensional torus $\mathbb{T}^2$. That is, there is a continuous splitting of the tangent bundle of $\mathbb{T}^2$ into a direct sum $E^u \oplus E^s$ which is preserved by the derivative DT and such that the unstable subbundle E^u is uniformly expanded by DT and the stable subbundle E^s is uniformly contracted by DT. Any such Anosov diffeomorphism T is homotopic and topologically conjugate to a hyperbolic toral automorphism L given by an integer matrix with determinant one and no eigenvalues of absolute value one. This was first proven by Franks in 1969 [6] under the assumption that all points on the torus are non-wandering (in fact, his result was for an n-dimensional torus). A year later Newhouse [22] pointed out that this assumption is satisfied when either $\dim E^s = 1$ or $\dim E^u = 1$, which provided the classification of Anosov diffeomorphisms up to topological conjugacy in dimensions 2 and 3. The case of dimension $n \geq 4$ was settled by Manning [18] in 1974.

Suppose T_1 and T_2 are two C^r $(r > 1)$ Anosov diffeomorphisms in the homotopy class of a fixed hyperbolic automorphism L. It follows from

2020 *Mathematics Subject Classification.* 37D35, 37B10, 37A60, 37C15, 37D20.
Key words and phrases. Anosov diffeomorphisms, smooth conjugacy problem, thermodynamic formalism, pressure function, equilibrium states, Hölder potentials.
T.K. is supported by grants from the Simons Foundation #430032.
A.Q. is supported by a grant from NSERC.

the above that there is a homeomorphism h such that $h \circ T_1 = T_2 \circ h$. The problem of determining when h has the same regularity as the maps T_1 and T_2 is known as the smooth conjugacy problem and has been studied extensively, see e.g. [14, 12, 7, 8]. Already in 1967 Anosov [1] constructed examples which showed that h may be merely Hölder even for highly regular T_1 and T_2, which initially discouraged further study of the problem (see comments in [25]). However, a series of papers [19, 15, 20, 16], authored, in various combinations, by de la Llave, Marco, and Moriyón, appeared in the 1980s focusing on the study of the conjugacy of C^∞ diffeomorphisms on $\mathbb{T}^2$. The culmination of their work is the following theorem.

Theorem. [16] *Let T_1 and T_2 be C^∞ Anosov diffeomorphisms of $\mathbb{T}^2$. If they are topologically conjugate and the Lyapunov exponents at corresponding periodic orbits are the same, then the conjugating homeomorphism is C^∞.*

Later it was shown that the equality of the corresponding Lyapunov exponents for C^r Anosov diffeomorphisms on $\mathbb{T}^2$ implies that the conjugacy is $C^{r-\epsilon}$, however it is no longer true on $\mathbb{T}^4$ even for C^∞ maps [17]. The case of $\mathbb{T}^3$ is still open, with a positive result recently obtained when one of the diffeomorphisms is an automorphism [5].

Note that if h is differentiable, then for any point x of period n for T_1, $h(x)$ is of period n for T_2 and

$$DT_1^n(x) = Dh^{-1}(h(x))DT_2^n(h(x))Dh(x).$$

We see that the Lyapunov exponents of x under T_1 and $h(x)$ under T_2 coincide. The result of [16] is quite remarkable since a condition, which is a priori weaker than h being C^1, is shown to imply that h is C^∞. F. Rodriguez Hertz asked whether we can get away with even less. He proposed to replace the assumption of equality of the Lyapunov exponents by the equality of the pressure functions of the geometric potentials.

To introduce the pressure function we first define the topological pressure using the variational principle. The topological pressure of a continuous potential $\phi : \mathbb{T}^2 \to \mathbb{R}$ with respect to a dynamical system $T : \mathbb{T}^2 \to \mathbb{T}^2$ is given by

$$P_{\rm top}(T, \phi) = \sup_\mu \left\{ h_\mu(T) + \int \phi \, d\mu \right\},$$

where μ runs over the set of all T-invariant probability measures on $\mathbb{T}^2$ and $h_\mu(T)$ is the measure-theoretic entropy of μ. A measure μ which realizes the supremum is called an equilibrium state of ϕ. By a celebrated result of Bowen [2], for an Anosov diffeomorphism T any Hölder potential $\phi : \mathbb{T}^2 \to \mathbb{R}$ has a unique equlibrium state μ_ϕ. Equilibrium states are mathematical generalizations of Gibbs distributions in statistical physics. The most important ones are the measure of maximal entropy, which is the equilibrium state of a constant potential, and the SRB measure, which is the equilibrium state of the *geometric potential*. The geometric potential is the negative logarithm of the Jacobian of T along the unstable bundle E^u,

$$\phi_T^u(x) = -\log \left| D_u T(x) \right|.$$

The *pressure function* of a potential ϕ is the map $t \mapsto P_{\mathrm{top}}(T, t\phi)$, where t is a real valued parameter. Information about various dynamical properties of an Anosov system is encoded into the pressure function of the geometric potential. For example, when T is area preserving, the positive Lyapunov exponent of T with respect to the normalized Lebesgue measure (which is the equilibrium state of ϕ_T^u) is given by the negative derivative of the pressure function of ϕ_T^u at $t = 1$, while the derivative at $t = 0$ gives the Lyapunov exponent with respect to the measure of maximal entropy of T. F. Rodriguez Hertz asked whether information on the regularity of the conjugating homeomorphism can also be extracted from the pressure functions of the geometric potentials of the corresponding maps. More precisely,

Question 1. [11, attr. F. Rodriguez Hertz] *Let T_1 and T_2 be C^∞ area-preserving Anosov diffeomorphisms on $\mathbb{T}^2$ that are homotopic. Assume $P_{\mathrm{top}}(T_1, t\phi_{T_1}^u) = P_{\mathrm{top}}(T_2, t\phi_{T_2}^u)$ for all t. Does this imply that T_1 and T_2 are C^∞ conjugate?*

We point out that the answer to the above question is positive when one of the diffeomorphisms is an automorphism. Indeed if T_1 is an automorphism, then $\phi_{T_1}^u$ is constant, so that $P_{\mathrm{top}}(T_1, t\phi_{T_1}^u)$, and hence $P_{\mathrm{top}}(T_2, t\phi_{T_2}^u)$ is affine. However, pressure functions of Hölder continuous functions are known to be strictly convex unless the underlying potential is cohomologous to a constant. Hence $\phi_{T_2}^u$ is cohomologous to the constant $\phi_{T_1}^u$. This guarantees that the Lyapunov exponents of periodic points of T_2 match those of periodic orbits of T_1, so that T_1 and T_2 are C^∞ conjugate by the above result.

One reason that Anosov diffeomorphisms on $\mathbb{T}^2$ are well-understood is that they admit symbolic codings. Using a Markov partition of $\mathbb{T}^2$ one can find a finite set $\mathcal{A}$ (indexing the set of rectangles of the Markov partition) and a mixing subshift of finite type $\Omega \subset \mathcal{A}^{\mathbb{Z}}$ such that there exists a finite-to-one factor map $\pi : \Omega \to \mathbb{T}^2$ which is Hölder. Then

$\phi_T^u \circ \pi$ is a Hölder potential on Ω. It turns out that in the symbolic setting, a related question to Question 1 has been studied by Pollicott and Weiss in [23].

Suppose (Ω, σ) is a subshift of finite type and $\psi : \Omega \to \mathbb{R}$ is a Hölder potential. Denote the Birkhoff sum of ψ by $S_n\psi(x) = \sum_{k=0}^{n-1} \psi(\sigma^k x)$. The multi-set $\{(S_n\psi(x), n) : \sigma^n x = x\}$ is called the *unmarked orbit spectrum of* ψ. In [23] the extent to which a potential is determined by its periodic orbit invariants such as its orbit spectrum and its pressure function was investigated. Note that for subshifts of finite type the pressure function can be defined topologically as

$$P_{\text{top}}(\sigma, t\psi) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\sigma^n x = x} e^{tS_n\psi(x)} \right),$$

and therefore any two potentials with the same unmarked orbit spectrum must have identical pressure functions. The converse is not true. It was shown by Pollicott and Weiss that there exists an uncountable family of Hölder continuous functions on a full shift with different unmarked orbit spectra, but all sharing the same pressure function.

Since for Anosov $T : \mathbb{T}^2 \to \mathbb{T}^2$ we have $-\log \left| D_u T^n \right| = S_n \phi_T^u$, the equality of the Lyapunov exponents at periodic orbits for torus diffeomorphisms T_1 and T_2 corresponds to the equality of the unmarked orbit spectra of their geometric potentials. Hence Question 1 may be seen as asking whether Hölder functions arising from geometric potentials of Anosov diffeomorphisms on the torus are special enough that the equality of their pressure functions implies the equality of their unmarked orbit spectrum. That turns out not to be the case.

We show that the set of pressure functions for Anosov diffeomorphisms with respect to their geometric potentials is equal to the set of pressure functions for the hyperbolic automorphism with respect to Hölder potentials.

Theorem 1. *Let L be a hyperbolic automorphism of $\mathbb{T}^2$ and let μ be the equilibrium state for a Hölder continuous potential ϕ with $P_{\text{top}}(L, \phi) = 0$. Then there exists a C^{1+H} area-preserving Anosov diffeomorphism T of $\mathbb{T}^2$ such that*

- *the system $T\colon (\mathbb{T}^2, \mathsf{Leb}) \to (\mathbb{T}^2, \mathsf{Leb})$ is conjugate to $L\colon (\mathbb{T}^2, \mu) \to (\mathbb{T}^2, \mu)$ by a map h;*
- *the potential $-\log |D_u T| \circ h$ is cohomologous to ϕ.*

In this theorem, and throughout the paper, we write T is C^{1+H} to mean that there exists $0 < \alpha < 1$ where T is $C^{1+\alpha}$.

A statement similar to the above theorem could be deduced from the work by Cawley [4] which establishes a bijection between Teichmüller space of an Anosov diffeomorphism and the quotient of Hölder functions by the subspace of coboundaries plus constants. However the proofs in [4] appear to be rather opaque. Our approach is constructive where the main step – the change of coordinates – is given by an explicit formula in terms of the equilibrium state of ϕ.

In view of Theorem 1, to solve Question 1 we need to find Hölder potentials having identical pressure functions with respect to an automorphism L, but different unmarked orbit spectra. From the work of Pollicott and Weiss one might expect uncountably many such potentials on the corresponding subshift of finite type. However, there is no reason to expect that any of these potentials will be Hölder continuous on the torus. Hence we have to employ another construction to produce torus continuous examples. We obtain

Theorem 2. *There exist homotopic C^{1+H} area-preserving Anosov diffeomorphisms T_1 and T_2 on $\mathbb{T}^2$ such that $P_{\rm top}(T_1, t\phi^u_{T_1}) = P_{\rm top}(T_2, t\phi^u_{T_2})$ for all t, but T_1 and T_2 fail to be C^1 conjugate.*

In fact our results give countably many homotopic Hölder differentiable area-preserving Anosov diffeomorphisms, none of which are C^1 conjugate, but all having the same pressure function. We do not know whether one can find uncountably many such maps, as would be suggested by the result in [23].

We remark that our examples, which are in the C^{1+H} category, do not directly respond to the C^∞ question of Rodriguez Hertz; however they strongly suggest a negative answer to that question also.

Acknowledgement. Part of this work was completed during our one-week stay at the *Centre International de Rencontres Mathématiques* in Luminy, France through the *Research in Residence* program. We thank CIRM for the support and hospitality.

2. Preliminary Results

2.1. **Gibbs Measures and Radon-Nikodym Derivative.**

In recent works an invariant measure is termed Gibbs if the weight of the Bowen balls of order n satisfies the growth estimate given in [2, Theorem 1.2]. We recall the original definition of a Gibbs state introduced by Ruelle [24] and Capocaccia [3], which is equivalent to Bowen's property from [2] in our situation. Let $T : M \to M$ be an expansive homeomorphism on a compact metric space M. A map χ from some open set $U \subset M$ into M is called *conjugating* for the system (M,T) if $d(T^n \circ \chi(x), T^n(x)) \to 0$ for $|n| \to \infty$ uniformly in $x \in U$. In the case of an Anosov automorphism L, the conjugating homeomorphisms are locally given by $x \mapsto x+v$ where v is homoclinic to 0. For this article, we only need the global conjugating homeomorphisms $x \mapsto x+v$.

Suppose ϕ is a continuous function on M. A probability measure μ on M is a *Gibbs state* for ϕ if for every conjugating homeomorphism $\chi : U \to \chi(U)$ where $U = U_\chi$ is an open set in M the measure $\chi_*(\mu|_U)$ is absolutely continuous with respect to $\mu|_{\chi(U)}$, with Radon-Nikodym derivative

$$\frac{d\chi_*\mu}{d\mu} = \exp \sum_{n\in\mathbb{Z}} \left[\phi \circ T^n \circ \chi^{-1} - \phi \circ T^n\right]. \tag{1}$$

For an axiom A diffeomorphism the equilibrium state of a Hölder potential ϕ is also a Gibbs state for ϕ, which is proven in Ruelle's book [24, Theorem 7.18]. A result of Haydn [9] is that the converse holds as well. In fact, Haydn and Ruelle show in [10] that equilibrium states and Gibbs states are equivalent for expansive homeomorphisms with specification and Bowen potentials.

We need the regularity properties of the Radon-Nikodym derivative (1). Although the question of regularity seems to be very natural, we were not able to locate a corresponding result in the literature. We provide a proof in the case of Anosov automorphisms, however the same argument can be straightforwardly generalized to Anosov diffeomorphisms, Axiom A diffeomorphisms or more general Smale spaces.

Lemma 3. *Let $L : \mathbb{T}^2 \to \mathbb{T}^2$ be an Anosov automorphism, let v be homoclinic to 0 and let $\tau(x) = x - v$. Let ϕ be a Hölder continuous function and let μ be the corresponding equilibrium state. Then the Radon-Nikodym derivative $\frac{d\tau_*\mu}{d\mu}$ in* (1) *above is Hölder continuous.*

Proof. Let λ be the expanding eigenvalue of L. Then there exist C_1 and C_2 such that $d(L^n v, 0) \le C_1\lambda^{-|n|}$ and $d(L^n x, 0) \le C_2\lambda^{|n|}d(x,0)$ for all $n \in \mathbb{Z}$. We let $C_3 > 0$ and $\alpha \in (0,1)$ be such that $|\phi(x) - \phi(y)| \le C_3 d(x,y)^\alpha$ for all $x, y \in \mathbb{T}^2$.

We define

$$\theta(x) = \sum_{n\in\mathbb{Z}} \left[\phi(L^n(x+v)) - \phi(L^n x)\right].$$

Suppose $x, y \in \mathbb{T}^2$ satisfy $d(x,y) < \lambda^{-2k}$ for some k. Then we calculate

$$\begin{aligned}|\theta(y)-\theta(x)| &\le \sum_{n\in\mathbb{Z}} \left|\phi(L^n(y+v)) - \phi(L^n y) - \phi(L^n(x+v)) + \phi(L^n x)\right| \\ &\le \sum_{|n|\le k} \left[|\phi(L^n(y+v)) - \phi(L^n(x+v))| + |\phi(L^n(y)) - \phi(L^n(x))|\right] \\ &\quad + \sum_{|n|>k} \left[|\phi(L^n(y+v)) - \phi(L^n(y))| + |\phi(L^n(x+v)) - \phi(L^n(x))|\right].\end{aligned}$$

We bound the sums by geometric series and obtain

$$|\phi(L^n y) - \phi(L^n x)| \le C_3(C_2\lambda^{|n|} d(x,y))^\alpha \le C_3 C_2^\alpha \lambda^{-(2k-|n|)\alpha},$$

with the same bound for $|\phi(L^n(y+v)) - \phi(L^n(x+v))|$. Likewise, $|\phi(L^n(x+v)) - \phi(L^n(x))| \le C_3 C_1^\alpha \lambda^{-|n|\alpha}$ with the same bound for $|\phi(L^n(y+v)) - \phi(L^n(y))|$. Summing the geometric series, we obtain $|\theta(y)-\theta(x)| \le K\lambda^{-k\alpha}$, where $K = 4C_3(C_1^\alpha + C_2^\alpha)/(1-\lambda^{-\alpha})$ showing that θ is Hölder as required. □

2.2. **Coding for Toral Automorphisms.** Let L be a mixing toral automorphism of $\mathbb{T}^2$ and we let $\mathcal{P}$ be a generating Markov partition, which we assume to consist of (closed) rectangles whose boundaries are pieces of the unstable and stable manifolds through the origin. We make the further assumption that if A and B are elements of the partition, then $(A+v)\cap B$ is connected (either a rectangle or an empty set). This condition is automatically satisfied if $\operatorname{diam}(\mathcal{P}) < \frac{1}{2}$, and so may be assumed without loss of generality by replacing $\mathcal{P}$ with a Markov partition of the form $\bigvee_{j=0}^{m-1} L^{-j}\mathcal{P}$ if necessary.

For $\mathcal{A} = \{0, ..., \#(\mathcal{P})-1\}$ let $\Omega \subset \mathcal{A}^{\mathbb{Z}}$ be the corresponding shift of finite type and let $\pi\colon \Omega \to \mathbb{T}^2$ be the corresponding finite-to-one factor map from (Ω,σ) to $(\mathbb{T}^2, L)$. The map π is one-to-one on a set of measure 1 with respect to any invariant measure on Ω. We equip Ω with the standard metric on Ω where $d(\omega,\omega') = 2^{-n}$ if $\omega_j = \omega'_j$ whenever $|j| < n$, but $\omega_{\pm n} \ne \omega'_{\pm n}$.

If ϕ is a Hölder continuous function on $\mathbb{T}^2$, we let μ be its equilibrium measure. We also set $\psi = \phi\circ\pi$ to be the corresponding potential on Ω and let ν be the equilibrium measure of ψ. Since π is one-to-one ν-almost everywhere, $\pi_*\nu = \mu$. Let $\Omega^+ \subset \mathcal{A}^{\mathbb{N}_0}$ be the one-sided version of Ω that is the image of Ω under the map $p_+\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{N}_0}$ defined by $p_+(\omega)_n = \omega_n$ for $n \ge 0$. Similarly, let $\Omega^- \subset \mathcal{A}^{-\mathbb{N}}$ be the image of Ω under the restriction map $p_-\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{-\mathbb{N}}$. Then $\nu^+ = (p_+)_*\nu$ and $\nu^- = (p_-)_*\nu$ are the measures corresponding to ν on Ω^+ and ω^- respectively.

The main symbolic result we are using is the local product structure of ν. Ruelle proves in [24, Lemma 5.9] that ν has *local product structure*, i.e. $d\nu(\omega) = \hat{\varrho}(\omega)\, d\hat{\nu}^+(p_+(\omega))\, d\hat{\nu}^-(p_-(\omega))$ where $\hat{\nu}^+$ is a probability measure on Ω^+, $\hat{\nu}^-$ is a probability measure on Ω^-, and $\hat{\varrho}$ is a positive continuous function on Ω. Furthermore, it is shown in [24, Lemma 5.23] that $\hat{\varrho}$ is Hölder on Ω, and the functions $\hat{\varrho}^+(\omega^+) = \int \hat{\varrho}(\omega)\, d\hat{\nu}^-(\omega^-)$, $1/\hat{\varrho}^+(\omega^+)$ are Hölder on Ω^+. Analogous statements hold for $\hat{\varrho}^-(\omega^-)$. Note that for each $\omega^+ \in \Omega^+$ the integral is taken over the set $\{\omega^- \in \Omega^- \colon \omega^-_{-1}\omega^+_0$ is legal in $\Omega\}$. In this case the measure ν^+ on Ω^+ is given by $d\nu^+ = \varrho^+(\omega^+)\, d\hat{\nu}^+$; similarly for ν^-.

We are mostly concerned with the structure of ν on the cylinder $[0] = \{\omega \in \Omega : \omega_0 = 0\}$. We let $A^- = \{\omega^- \in \Omega^- \colon \omega^-_{-1}0$ is legal in $\Omega\}$. For $\omega^- \in A^-$ and $\omega^+ \in p_+([0])$ we write $\varrho^+(\omega^+) = \int_{A^-} \hat{\varrho}(\omega^-\omega^+)\, d\hat{\nu}^-(\omega^-)$, $\varrho^-(\omega^-) = \int_{[0]} \hat{\varrho}(\omega^-\omega^+)\, d\hat{\nu}^+(\omega^+)$, and $\varrho(\omega^-\omega^+) = \frac{\hat{\varrho}(\omega^-\omega^+)}{\varrho^-(\omega^-)\varrho^+(\omega^+)}$, so that $d\nu(\omega) = \rho(\omega)\, d\nu^+(\omega^+)\, d\nu^-(\omega^-)$. In particular,

$$
\begin{aligned}
\int_{A^-} \varrho(\omega^-\omega^+)\, d\nu^-(\omega^-) &= \int_{A^-} \frac{\hat{\varrho}(\omega^-\omega^+)}{\varrho^-(\omega^-)\varrho^+(\omega^+)}\, d\nu^-(\omega^-) \\
&= \frac{1}{\varrho^+(\omega^+)} \int_{A^-} \hat{\varrho}(\omega^-\omega^+)\, d\hat{\nu}^-(\omega^-) \\
&= 1
\end{aligned}
\tag{2}
$$

We summarize the above in the following lemma which is frequently used throughout this article.

Lemma 4 (Ruelle [24]). *Let ψ be a Hölder continuous function on a mixing shift of finite type Ω and let ν be its equilibrium state. Then ν has* local product structure*. That is, on the cylinder set* $[0]$ *there exist a positive Hölder continuous function $\varrho(\omega)$ such that $d\nu(\omega) = \varrho(\omega)\, d\nu^+(\omega^+)\, d\nu^-(\omega^-)$ where ν^-, ν^+ are the restrictions of ν to Ω^+, Ω^- respectively, and ω denotes the concatenation of ω^- and ω^+.*

It is shown by Walters in [26] that under the assumptions of the above lemma there is a Hölder function $g : \Omega^+ \to (0, 1)$ such that $\log g$ is cohomologous to ϕ and ν^+ is the unique g-measure for g, i.e. for $\omega^+ \in \Omega^+$

$$
g(\omega^+) = \lim_{\substack{\operatorname{diam}(S)\to 0 \\ \nu^+(S)\neq 0,\, \omega^+\in S}} \frac{\nu^+(S)}{\nu^+(\sigma_+(S))}. \tag{3}
$$

Since the map $\pi \colon \Omega \to \mathbb{T}^2$ is Hölder continuous, given a Hölder continuous function ϕ on the torus, we see that $\phi \circ \pi$ is Hölder; however many Hölder continuous functions on the shift cannot be written in the form $\phi \circ \pi$. We call a function f defined on Ω *torus-Hölder* if it can be written in the form $\phi \circ \pi$ where ϕ is a Hölder continuous function of the torus. A subset R of Ω is called a *rectangle* if it satisfies the following conditions

- $\omega, \omega' \in R$ implies the concatenation $p_-(\omega)p_+(\omega')$ belongs to R;
- $\pi(R)$ is connected;
- $\operatorname{diam}(\pi(R)) < \frac{1}{2}$;
- $R = \pi^{-1}(\pi(R))$.

Lemma 5. *Let L be an Anosov automorphism of $\mathbb{T}^2$ and let $\mathcal{P}$ be a Markov partition as described above. Let Ω be the corresponding shift of finite type and let $\pi\colon \Omega \to \mathbb{T}^2$ be the natural factor map. Let R be a rectangular subset of a cylinder set $[i]$ in Ω and suppose that $f\colon R \to \mathbb{R}$ is a Hölder continuous function. If f has the property that $f(\omega) = f(\omega')$ whenever $\pi(\omega) = \pi(\omega')$, then f may be expressed as $h \circ \pi$ where h is a Hölder continuous function defined on $\pi(R) \subset \mathbb{T}^2$.*

Proof. Since $f(\omega) = f(\omega')$ when $\pi(\omega) = \pi(\omega')$, we see that f takes the same value on each element of $\pi^{-1}(x)$ for any $x \in \pi(R)$. Hence $h(x) := f(\pi^{-1}x)$ is well-defined on the rectangle $A := \pi(R)$ which has sides parallel to the stable and unstable directions. Since f is Hölder continuous, let c and α be such that $|f(\omega) - f(\omega')| \le c\alpha^n$ whenever $d(\omega, \omega') \le 2^{-n}$.

Since A is a rectangle in $\mathbb{T}^2$, we define for $x, y \in A$, $[\![x, y]\!]_A$ to be the unique point z in A such that the line segments $[x, z]$ and $[z, y]$ lie in A with $[x, z]$ in the stable direction and the $[z, y]$ in the unstable direction. We now estimate $|h(x) - h(z)|$. An exactly similar estimate applies to $|h(z) - h(y)|$. Let C be the constant (depending only on the angle between the stable and unstable directions) so that if x, y lie in A then $d(x, [\![x, y]\!]_A), d(y, [\![x, y]\!]_A) \le Cd(x, y)$. Let λ be the expanding eigenvalue and let n be the smallest natural number such that $C^{-1} \operatorname{diam}(\mathcal{P})\lambda^{-n} \le d(x, y)$.

Let $x = \pi(\xi)$ and $[\![x, y]\!]_A = \pi(\zeta)$. Then either x and $[\![x, y]\!]_A$ lie in the same element of $L^j\mathcal{P}$ for each $0 \le j < n$, in which case $|h(x) - h([\![x, y]\!]_A)| = |f(\xi) - f(\zeta)| \le c\alpha^n$ or there exists a point w in $\partial L^{-(n-1)}\mathcal{P} \cap [x, [\![x, y]\!]_A]$. Since $d(x, w)$ and $d([\![x, y]\!]_A, w)$ are less than $\operatorname{diam}(\mathcal{P})\lambda^{-(n-1)}$ and w is on the boundary, x and w must belong to a common element of $L^{-(n-1)}\mathcal{P}$ and similarly for w and $[\![x, y]\!]_A$, see Fig 1.

Now write $w = \pi(\eta) = \pi(\eta')$ where $\eta_{-\infty}^{n-1} = \xi_{-\infty}^{n-1}$ and ${\eta'}_{-\infty}^{n-1} = \zeta_{-\infty}^{n-1}$. We then have $|h(x) - h(z)| = |f(\xi) - f(\zeta)| \le |f(\xi) - f(\eta)| + |f(\eta') - f(\zeta)| \le 2c\alpha^n$, where we made use of the fact that $f(\eta) = f(\eta')$. Combining this with the analogous estimate for $|h([\![x, y]\!]_A) - h(y)|$, we see

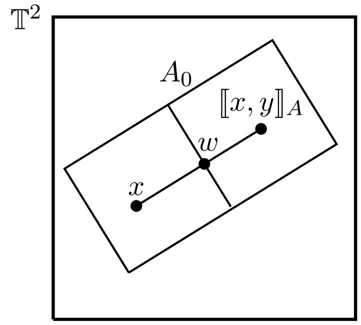

FIGURE 1. On $\mathbb{T}^2$ the unstable and stable directions are shown as north-east and north-west respectively.

$|h(x)-h(y)| \le 4c\alpha^n \le 4c\big(Cd(x,y)/(\lambda\,\mathrm{diam}(\mathcal{P})\big)^{-\log\alpha/\log\lambda}$, so that h is Hölder as required. $\square$

3. Anosov realization

In this section we show that given a hyperbolic automorphism L for any positive Hölder continuous potential ϕ with zero topological pressure there exists a conjugate Anosov diffeomorphism T for which the geometric potential is cohomologous to ϕ.

Theorem 6. *Let L be an Anosov automorphism of $\mathbb{T}^2$ and let μ be the equilibrium state for a Hölder continuous potential ϕ with $P_{\mathrm{top}}(L,\phi)=0$. Then there exists a C^{1+H}-atlas on $\mathbb{T}^2$ with respect to which L is an Anosov diffeomorphism with Hölder derivative and its geometric potential is cohomologous to ϕ.*

We prove the theorem in a number of steps.

3.1. **Definition of new C^{1+H} atlas.** We let $\mathcal{H}$ denote the collection of points of $\mathbb{T}^2$ that are homoclinic to 0 under the action of L. Since L is an automorphism, it follows that if $v\in\mathcal{H}$ and $x\in\mathbb{T}^2$ then $d(L^n(x+v),L^n(x))=d(L^nv,0)\to 0$ as $|n|\to\infty$. Recall that the points homoclinic to 0 are dense in $\mathbb{T}^2$ (see e.g. [21]).

For the remainder of this section A_0 denotes the element of the partition $\mathcal{P}$ which corresponds to the cylinder $[0]$ in Ω, i.e. $\pi([0])=A_0$.

Lemma 7. *Let $w\in\mathcal{H}$ and suppose that $A_0\cap(A_0-w)$ has non-empty interior. Then there exist vectors $u,v\in\mathcal{H}$ such that:*

- *$u+v=w$;*
- *if $x\in Int(A_0\cap(A_0-u))$ then the line segment $[x,x+u]$ lies in $Int(A_0)$ and is parallel to the stable direction;*

- *if $x \in Int(A_0 \cap (A_0 - v))$ then the line segment $[x, x+v]$ lies in $Int(A_0)$ and is parallel to the unstable direction;*
- *$Int(A_0 \cap (A_0 - w)) = Int\big(A_0 \cap (A_0 - u) \cap (A_0 - v)\big)$.*

Proof. For any $x \in \mathrm{Int}(A_0 \cap (A_0 - w))$, since A_0 is a parallelogram with edges parallel to the stable and unstable directions, the vector w may be expressed as a sum of pieces u and v parallel to the stable and unstable directions, where $[x, x+u]$ and $[x, x+v]$ lie in A_0. Note that $x+u$ is the point of intersection of the stable manifold of x and the unstable manifold of $x+w$. Linearity of L implies that u belongs to the stable manifold of 0 and unstable manifold of w. Since $w \in \mathcal{H}$, we conclude that $u \in \mathcal{H}$ as well. Similarly, $x+v$ is the point of intersection of the unstable manifold of x and the stable manifold of $x+w$, so $v \in \mathcal{H}$. □

We define two functions ξ_1 and ξ_2 on A_0. Let $\xi_1(x)$ be the μ-measure of the rectangle contained in A_0 lying to the left of the connected portion of the stable manifold of x within A_0 as illustrated in Figure 2. Similarly, let $\xi_2(x)$ be the μ-measure of the rectangle contained in A_0 lying below the connected portion of the unstable manifold of x within A_0. We denote $\xi(x) = (\xi_1(x), \xi_2(x))$.

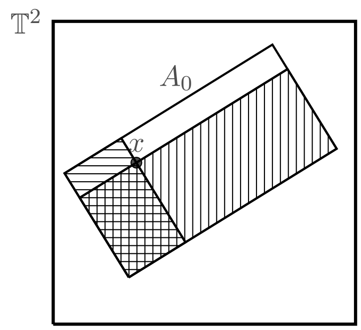

FIGURE 2. $\xi_1(x)$ is the measure of the region shaded with horizontal lines; $\xi_2(x)$ is the measure of the region shaded with vertical lines.

We introduce a new family of charts on $\mathbb{T}^2$. For $v \in \mathcal{H}$, let τ_v denote the translation $\tau_v(x) = x + v$. We then define a chart α_v with domain $\mathrm{Int}(A_0) - v$ by $\alpha_v = \xi \circ \tau_v$. Since $\mathcal{H}$ is dense in $\mathbb{T}^2$, the collection of charts covers all of $\mathbb{T}^2$. Our goal for the reminder of this subsection is to show that the family of charts $\{(\alpha_v, \mathrm{Int}(A_0) - v)\}_{v \in \mathcal{H}}$ forms a C^{1+H}-differentiable atlas on $\mathbb{T}^2$. We first prove a key lemma.

Let $v \in \mathcal{H}$ be such that $A_0 \cap (A_0 - v)$ has non-empty interior and such that for any $x \in \mathrm{Int}(A_0 \cap (A_0 - v))$, the line segment joining x and $x+v$ lies in $\mathrm{Int}(A_0)$ and is parallel to the unstable direction. Using the notation from Section 2.2 we consider the function $\xi_1(\pi(\omega))$ defined on $\pi^{-1}\big(\mathrm{Int}(A_0 \cap (A_0 - v))\big) \subset [0]$ in Ω and study the limit

$$\ell(\omega) := \lim_{\omega' \to \omega} \frac{\xi_1(\pi(\omega') + v) - \xi_1(\pi(\omega) + v)}{\xi_1(\pi(\omega')) - \xi_1(\pi(\omega))}. \tag{4}$$

Here the limit is taken over those ω' such that $\xi_1(\pi(\omega')) \neq \xi_1(\pi(\omega))$, that is those ω' such that $\pi(\omega')$ does not lie in the same local stable manifold as $\pi(\omega)$. This is illustrated in Figure 3.

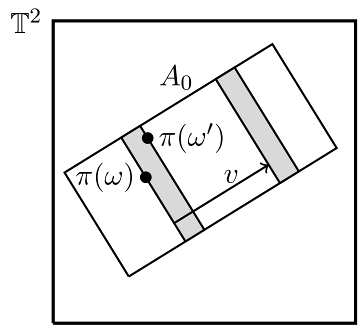

FIGURE 3. The numerator and denominator in the limit are respectively the measures of the right and left shaded rectangles.

Lemma 8. *Let $v \in \mathcal{H}$ be as described above. Then the limit $\ell(\omega)$, defined above, exists for all ω in $\pi^{-1}\big(\mathrm{Int}(A_0 \cap (A_0 - v))\big)$ and the function $\ell(\omega)$ is torus-Hölder on its domain.*

Proof. Letting $R[\omega, \omega']$ be the rectangle bounded on the top and bottom by the boundary of A_0 and the left and right by the stable manifolds through $\pi(\omega)$ and $\pi(\omega')$, we see that

$$\frac{\xi_1(\pi(\omega') + v) - \xi_1(\pi(\omega) + v)}{\xi_1(\pi(\omega')) - \xi_1(\pi(\omega))} = \frac{\mu(R[\omega, \omega'] + v)}{\mu(R[\omega, \omega'])} \tag{5}$$

We now apply the discussion of Section 2.1 to the case when T is the toral automorphism L. For any $v \in \mathbb{T}^2$ homoclinic to 0 under L the map $x \mapsto x + v$ is a (global) conjugating homeomorphism of $\mathbb{T}^2$. It follows from Lemma 3 that for an equilibrium state μ of a Hölder potential ϕ we have

$$\frac{d\mu(x+v)}{d\mu(x)} = \theta_v(x),$$

where

$$\theta_v(x) = \exp\left(\sum_{n \in \mathbb{Z}} \big[\phi(L^n(x+v)) - \phi(L^n(x))\big] \right). \tag{6}$$

Recall that by Lemma 3 the function $\theta_v\colon \mathbb{T}^2 \to \mathbb{R}$ is Hölder continuous.

We can now rewrite (4) as

$$\begin{aligned}\ell(\omega) &= \lim_{\omega'\to\omega} \frac{\mu(R[\omega,\omega'] + v)}{\mu(R[\omega,\omega'])}\\ &= \lim_{\omega'\to\omega} \frac{\int_{R[\omega,\omega']} \theta_v(x)\, d\mu(x)}{\int_{R[\omega,\omega']} 1\, d\mu(x)}.\end{aligned}$$

We observe that $\pi^{-1}R[\omega,\omega']$ is a subset of Ω consisting of points ζ such that ζ_0^∞ are the non-negative coordinates of points lying between $\pi(\omega)$ and $\pi(\omega')$. There is no restriction on the negative coordinates other than that $\zeta \in \Omega$ and $\zeta_0 = 0$. Write $A^+[\omega,\omega']$ for $\{\zeta^+ \in \Omega^+\colon \zeta^+$ are the non-negative coordinates of a point in $R[\omega,\omega']\}$ and A^- for $\{\zeta^- \in \Omega^-\colon\ \zeta^-_{-1}0$ is legal in $\Omega\}$. We now apply Lemma 4, giving

$$\ell(\omega) = \lim_{\omega'\to\omega} \frac{\int_{A^-}\int_{A^+[\omega,\omega']} \varrho(\zeta)\theta_v(\pi(\zeta))\, d\nu^+(\zeta^+)\, d\nu^-(\zeta^-)}{\int_{A^-}\int_{A^+[\omega,\omega']} \varrho(\zeta)\, d\nu^+(\zeta^+)\, d\nu^-(\zeta^-)}. \tag{7}$$

Since ϱ and $\theta_v \circ \pi$ are continuous, the integrands in the numerator and denominator may be approximated for ω' close to ω by $\varrho(\zeta^-\omega^+)\theta_v(\pi(\zeta^-\omega^+))$ and $\varrho(\zeta^-\omega^+)$ respectively. Since these new integrands don't depend on ζ^+, the inner integrals of the approximation to (7) are just the product of the integrand and $\nu^+(A^+[\omega,\omega'])$. Since ρ is strictly positive, cancelling the common factor, we now see that the limit exists, and

$$\begin{aligned}\ell(\omega) &= \frac{\int_{A^-} \varrho(\zeta^-\omega^+)\theta_v(\pi(\zeta^-\omega^+))\, d\nu^-(\zeta^-)}{\int_{A^-} \varrho(\zeta^-\omega^+)\, d\nu^-(\zeta^-)}\\ &= \int_{A^-} \varrho(\zeta^-\omega^+)\theta_v(\pi(\zeta^-\omega^+))\, d\nu^-(\zeta^-),\end{aligned}$$

where the second equality follows from (2). Further, since ϱ and $\theta_v \circ \pi$ are Hölder continuous functions on Ω, we can see that $\ell(\omega)$ is a Hölder continuous function of ω on $[0]$, depending only on the non-negative coordinates of ω.

In order to show that $\ell(\omega)$ is also torus-continuous, we consider ω belonging to the stable manifold of 0 (so that $\pi(\omega)$, which we assumed to lie in $\mathrm{Int}(A_0)$, lies on the boundary of two elements of $L^{-j}\mathcal{P}$ for some $j > 0$: one on the left and one on the right). In this case, $p_+^{-1}(\pi(\omega))$ consists of two elements, say ω^+ and η^+. We will show that $\ell(\omega^-\omega^+) = \ell(\omega^-\eta^+)$.

It will be convenient to find another expression for $\ell(\omega)$ in which $\pi(\omega)$ is translated by another homoclinic vector $\tilde{v}$ (which by Lemma 7 we can assume to be parallel to the unstable direction and to satisfy

$[\pi(\omega), \pi(\omega) + \tilde{v}] \subset \mathrm{Int}(A_0))$. Let $\tilde{R}[\omega, \omega'] = R[\omega, \omega'] + \tilde{v}$ and denote by $\tilde{A}^+[\omega, \omega']$ the set of future codes of points in the rectangle $\tilde{R}[\omega, \omega']$.

By similar arguments to those above and using the fact that $\theta_{v-\tilde{v}}(x) = \theta_{-\tilde{v}}(x)\theta_v(x - \tilde{v})$ which is immediate from the expression of the Radon-Nikodym derivative (6), we obtain

$$\begin{aligned}\ell(\omega) &= \lim_{\omega' \to \omega} \frac{\mu(\tilde{R}[\omega, \omega'] + v - \tilde{v})}{\mu(\tilde{R}[\omega, \omega'] - \tilde{v})} \\ &= \lim_{\omega' \to \omega} \frac{\int_{A^-} \int_{\tilde{A}^+[\omega,\omega']} \varrho(\zeta)\theta_{-\tilde{v}}(\pi(\zeta))\theta_v(\pi(\zeta) - \tilde{v})\, d\nu^+(\zeta^+)\, d\nu^-(\zeta^-)}{\int_{A^-} \int_{\tilde{A}^+[\omega,\omega']} \varrho(\zeta)\theta_{-\tilde{v}}(\pi(\zeta))\, d\nu^+(\zeta^+)\, d\nu^-(\zeta^-)},\end{aligned}$$

As before, taking a limit as ω' approaches ω, we see that

$$\ell(\omega^-\omega^+) = \frac{\int_{A^-} \varrho(\zeta^-\tilde{\omega}^+)\theta_{-\tilde{v}}(\pi(\zeta^-\tilde{\omega}^+))\theta_v(\pi(\zeta^-\omega^+))\, d\nu^-(\zeta^-)}{\int_{A^-} \varrho(\zeta^-\tilde{\omega}^+)\theta_{-\tilde{v}}(\pi(\zeta^-\tilde{\omega}^+))\, d\nu^-(\zeta^-)}, \tag{8}$$

where $\tilde{\omega}^+$ is the future coding of $\pi(\omega) + \tilde{v}$ corresponding to ω^+.

Letting $\tilde{\eta}^+$ be the future coding of $\pi(\omega) + \tilde{v}$ corresponding to η^+ we get

$$\begin{aligned}\ell(\omega^-\eta^+) &= \frac{\int_{A^-} \varrho(\zeta^-\tilde{\eta}^+)\theta_{-\tilde{v}}(\pi(\zeta^-\tilde{\eta}^+))\theta_v(\pi(\zeta^-\eta^+))\, d\nu^-(\zeta^-)}{\int_{A^-} \varrho(\zeta^-\tilde{\eta}^+)\theta_{-v'}(\pi(\zeta^-\tilde{\eta}^+))\, d\nu^-(\zeta^-)} \\ &= \frac{\int_{A^-} \varrho(\zeta^-\tilde{\eta}^+)\theta_{-\tilde{v}}(\pi(\zeta^-\tilde{\omega}^+))\theta_v(\pi(\zeta^-\omega^+))\, d\nu^-(\zeta^-)}{\int_{A^-} \varrho(\zeta^-\tilde{\eta}^+)\theta_{-\tilde{v}}(\pi(\zeta^-\tilde{\omega}^+))\, d\nu^-(\zeta^-)},\end{aligned} \tag{9}$$

where we used the facts $\pi(\zeta^-\tilde{\eta}^+) = \pi(\zeta^-\tilde{\omega}^+)$ and $\pi(\zeta^-\eta^+) = \pi(\zeta^-\omega^+)$.

Comparing (8) and (9), we see that the only place where they differ is that in the numerator and denominator, $\varrho(\zeta^-\tilde{\omega}^+)$ is replaced by $\varrho(\zeta^-\tilde{\eta}^+)$. However if $\tilde{v}$ is chosen so that $\pi(\omega) + \tilde{v}$ does not lie on the stable boundary of any element of $\bigvee_{0 \le j < n} L^{-j}\mathcal{P}$, then $\tilde{\eta}^+$ and $\tilde{\omega}^+$ agree for at least n symbols. Since ϱ is Hölder continuous, $\varrho(\zeta^-\tilde{\eta}^+)/\varrho(\zeta^-\tilde{\omega}^+)$ is uniformly exponentially close to 1 as ζ^- runs over A^-. It follows that $\ell(\omega) = \ell(\omega^-\zeta^+)$, so that ℓ is torus-continuous. □

We are now ready to establish that the atlas $\{(\alpha_v, \mathrm{Int}(A_0 - v)) \colon v \in \mathcal{H}\}$ is C^{1+H}. We need to prove that for $v_0, v_1 \in \mathcal{H}$ with the property that $\mathrm{Int}(A_0 - v_0) \cap \mathrm{Int}(A_0 - v_1) \neq \emptyset$, the map $\alpha_{v_1} \circ \alpha_{v_0}^{-1}$ is differentiable with Hölder continuous derivative. In this case, observe $\alpha_{v_1} \circ \alpha_{v_0}^{-1} = (\xi \circ \tau_{v_1}) \circ (\xi \circ \tau_{v_0})^{-1} = \xi \circ \tau_w \circ \xi^{-1}$, where $w = v_1 - v_0 \in \mathcal{H}$.

Using Lemma 7, we write $w = v + u$, where v is in the unstable direction and u is in the stable direction. Moreover, if both x and $x + w$ are in $\mathrm{Int}(A_0)$, then the line segment joining x and $x + v$ lies in $\mathrm{Int}(A_0)$, so that v satisfies the conditions of Lemma 8. Let h_1 be the

Hölder continuous function on $\mathrm{Int}(A_0)\cap\mathrm{Int}(A_0-w)$ such that $\ell=h_1\circ\pi$ on their domain.

We now evaluate the derivative of $\xi\circ\tau_w\circ\xi^{-1}$ using the function ℓ. If (a,b) and (a,b') have the same first coordinate and are in the range of $\xi\circ\tau_w\circ\xi^{-1}$, then we see from the definition of ξ that $\tau_w\circ\xi^{-1}(a,b)$ and $\tau_w\circ\xi^{-1}(a,b')$ lie on the same stable manifold, so that the first coordinates of $\xi\circ\tau_w\circ\xi^{-1}(a,b)$ and $\xi\circ\tau_w\circ\xi^{-1}(a,b')$ agree. Similarly the second coordinates of $\xi\circ\tau_w\circ\xi^{-1}(a,b)$ and $\xi\circ\tau_w\circ\xi^{-1}(a',b)$ agree, so that $\xi\circ\tau_w\circ\xi^{-1}(a,b)$ is of the form $(f_1(a),f_2(b))$. We see from the definition of ℓ that for (a,b) in the domain of $\xi\circ\tau_w\circ\xi^{-1}$, $f_1'(a)=h_1(\xi^{-1}(a,b))=h_1(\xi_1^{-1}(a)\cap\xi_2^{-1}(b))$. Since h_1 is constant on local stable manifolds, this can also be written as $h_1(\xi_1^{-1}(a))$.

We verify that f_1' is Hölder; an almost identical argument will show that f_2' is Hölder. Let e_u be the unit unstable direction and z be the bottom left corner of A_0. Using $\iota(t)=\xi_1(z+te_u)$ we can write $f_1'(a)=h_1(z+\kappa^{-1}(a)e_u)$. To show that f_1' is Hölder, it therefore suffices to show that κ^{-1} is Hölder, which follows from an estimate of the form $|\kappa(t')-\kappa(t)|\geq c|t-t'|^\beta$. We conclude the proof by establishing an estimate of this form. Let $t'>t$ and let n be such that $|t-t'|\geq 2\,\mathrm{diam}(\mathcal{P})\lambda^{-n}$ (as before, λ denotes the expanding eigenvalue of the matrix defining L). Then between the local stable manifolds through $z+t\,e_u$ and $z+t'\,e_u$, there is at least one full element of $\bigvee_{j=0}^{n-1}L^{-j}\mathcal{P}$. By the Gibbs inequality, these elements have measure at least $c'e^{-\delta n}$ for some c' and δ that are independent of t and t', so that $|\kappa(t')-\kappa(t)|\geq c'e^{-\delta n}$. But from the bound on $|t-t'|$, we deduce $|\kappa(t')-\kappa(t)|\geq c|t-t'|^\beta$ for some c and β as required.

3.2. **Differentiability of L with respect to the new atlas.** We proved in Section 3.1 that the family of charts $\Xi=\{(\alpha_v,\mathrm{Int}(A_0)-v)\}_{v\in\mathcal{H}}$ form a C^{1+H}-differentiable atlas on $\mathbb{T}^2$. In this section we show that $L:(\mathbb{T}^2,\Xi)\to(\mathbb{T}^2,\Xi)$ is C^{1+H}.

We first consider the case when $A_0\cap L^{-1}A_0$ has non-empty interior. We claim that it suffices to establish that $\xi\circ L\circ\xi^{-1}$ is C^{1+H} on $\xi(A_0\cap L^{-1}A_0)$. To see this, let $v_0,v_1\in\mathcal{H}$ be such that the domain of $\alpha_{v_1}\circ L\circ\alpha_{v_0}^{-1}$, i.e. $U:=(A_0-v_0)\cap L^{-1}(A_0-v_1)$, has non-empty interior. Let $(a,b)\in\alpha_{v_0}(U)$ and write $(a,b)=\alpha_{v_0}(x)=\xi(v_0+x)$. Let $w\in\mathcal{H}$ be such that $x+v_0+w\in\mathrm{Int}(A_0\cap L^{-1}A_0)$. We now see that on a neighbourhood of (a,b)

$$\begin{aligned}\alpha_{v_1}\circ L\circ\alpha_{v_0}^{-1}&=\xi\circ\tau_{v_1}\circ L\circ\tau_{-v_0}\circ\xi^{-1}\\&=(\xi\circ\tau_{v_1-Lv_0-Lw}\circ\xi^{-1})\circ(\xi\circ L\circ\xi^{-1})\circ(\xi\circ\tau_w\circ\xi^{-1}):\end{aligned}$$

$\xi\circ\tau_w\circ\xi^{-1}(a,b) = \xi(x+v_0+w)$; $\xi\circ L\circ\xi^{-1}(\xi(x+v_0+w)) = \xi(Lx+Lv_0+Lw) \in \xi(A_0)$; $\xi\circ\tau_{v_1-Lv_0-Lw}\circ\xi^{-1}(\xi(Lx+Lv_0+Lw)) = \xi(Lx+v_1) = \alpha_{v_1}\circ L\circ\alpha_{v_0}^{-1}(a,b)$. Once we establish that $\xi\circ L\circ\xi^{-1}$ is C^{1+H} on $\xi(A_0\cap L^{-1}A_0)$, it will follow from the results of the previous section that $\alpha_{v_1}\circ L\circ\alpha_{v_0}^{-1}$ is C^{1+H} on a neighbourhood of (a,b).

A similar argument to that in Section 3.1 shows that $\xi\circ L\circ\xi^{-1}(c,d)$ is of the form $(f_1(c), f_2(d))$ on its domain. We establish Hölder differentiability of $\xi\circ L\circ\xi^{-1}$ following the strategy of the previous section: first we show that f_1' is shift-Hölder and then we verify that f_1' is torus-continuous. We compute

$$f_1'(a) = \lim_{h\to 0}\frac{\xi\circ L\circ\xi^{-1}(a+h,b) - \xi\circ L\circ\xi^{-1}(a,b)}{h}. \tag{10}$$

From the definition of ξ we see that h is the μ-measure of the rectangle in A_0 lying between the stable manifolds through x and $x' = \xi^{-1}(a+h,b)$. Assuming that h is small enough that x' is also in $A_0\cap L^{-1}A_0$, we can write the numerator in the limit (10) as the μ-measure of the rectangle in A_0 lying between the stable manifolds through $L(x)$ and $L(x')$. We provide an illustration in Figure 4 below.

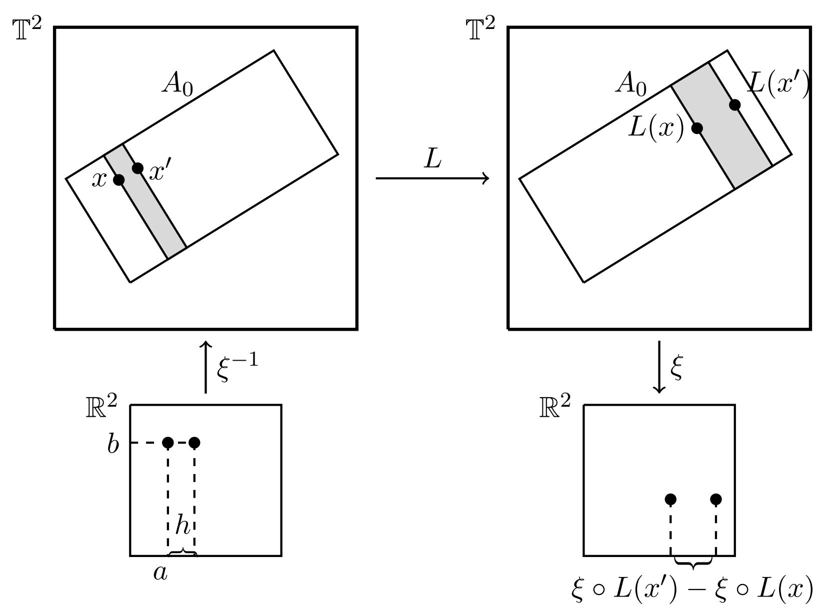

FIGURE 4. The μ-measures of the shaded rectangles on the right and left are the numerator and the denominator in the limit (10) respectively.

The derivative of f_1 can be represented symbolically on $[00] \subset \Omega$ as

$$\ell(\omega) = \lim_{\omega' \to \omega} \frac{\mu(R[\sigma(\omega), \sigma(\omega')])}{\mu(R[\omega, \omega'])},$$

where, as before, $R[\omega, \omega']$ and $R[\sigma(\omega), \sigma(\omega')]$ are the rectangles bounded on the top and bottom by the boundary of A_0 and on the sides by the stable manifolds through $\pi(\omega)$, $\pi(\omega')$ and $L(\pi(\omega))$, $L(\pi(\omega'))$ respectively. Again, we observe that $\pi^{-1}(R[\omega, \omega']) = A^- \times A^+[\omega, \omega']$ and $\pi^{-1}(R[\sigma(\omega), \sigma(\omega')]) = A^- \times A^+[\omega, \omega']$ where A^-, $A^+[\omega, \omega']$ are defined as in Section 3.1. On the other hand, $\pi^{-1}R[\sigma(\omega), \sigma(\omega')]$ is a subset of Ω consisting of points ζ such that ζ_0^∞ are the non-negative coordinates of points in $L(R[\omega, \omega'])$ and there are no additional restrictions on the negative coordinates. Using Lemma 4 we obtain

$$\ell(\omega) = \lim_{\omega' \to \omega} \frac{\nu(A^- \times \sigma_+(A^+[\omega, \omega']))}{\nu(A^- \times A^+[\omega, \omega'])} = \lim_{\omega' \to \omega} \frac{\nu^+(\sigma_+(A^+[\omega, \omega']))}{\nu^+(A^+[\omega, \omega'])}.$$

Since $\operatorname{diam}(A^+[\omega, \omega']) \to 0$ as $\omega' \to \omega$ and $\omega \in A^+[\omega, \omega']$, we conclude that $\ell(\omega) = \frac{1}{g(p_+(\omega))}$, where g is the g-function for measure ν^+. Since g is strictly positive and Hölder on Ω^+, ℓ is Hölder on Ω.

To prove that ℓ is torus continuous suppose that $x = \pi(\omega)$ lies on the stable manifold boundary of two elements of the partition $L^{-j}\mathcal{P}$ for some $j \in \mathbb{N}$. Let $\omega^-\omega^+$ and $\omega^-\eta^+$ be two different symbolic representations of x. To show that $\ell(\omega^-\omega^+) = \ell(\omega^-\eta^+)$ we apply the same steps as in Section 3.1. For any $N \in \mathbb{N}$, let $v \in \mathcal{H}$ be parallel to the unstable direction satisfying $x + v \in \operatorname{Int}(A_0 \cap L^{-1}A_0)$ and $x + v \notin \bigcup_{|k|<N} \partial L^k\mathcal{P}$. Let $\tilde{R}[\omega, \omega'] = R[\omega, \omega'] + v$ and denote by $\tilde{A}^+[\omega, \omega']$ the set of future coordinates of points in $\tilde{R}[\omega, \omega']$. Using the expression for the Radon-Nikodym derivative (6) we obtain

$$\mu(R[\omega, \omega']) = \int_{A^-} \int_{\tilde{A}^+[\omega, \omega']} \varrho(\zeta^-\zeta^+)\theta_{-v}(\pi(\zeta^-\zeta^+))\, d\nu^+(\zeta^+)\, d\nu^-(\zeta^-).$$

Similarly, let $\tilde{R}[\sigma(\omega), \sigma(\omega')] = R[\sigma(\omega), \sigma(\omega')] + L(v)$ and obtain

$$\mu(R[\sigma(\omega), \sigma(\omega')]) = \int_{A^-} \int_{\sigma^+(\tilde{A}^+[\omega, \omega'])} \varrho(\zeta^-\zeta^+)\theta_{-L(v)}(\pi(\zeta^-\zeta^+))\, d\nu^+\, d\nu^-.$$

Consider $\omega = \omega^-\omega^+$ and denote be $\tilde{\omega}^+$ the corresponding future coding of $\pi(\omega) + v$. By continuity of ϱ and θ for each ζ^- the inner integral of $\mu(R[\omega, \omega'])$ is approximately $\varrho(\zeta^-\tilde{\omega}^+)\theta_{-v}(\pi(\zeta^-\sigma^+(\tilde{\omega}^+)))\nu^+(\tilde{A}^+[\omega, \omega'])$ and similarly the inner integral of $\mu(R[\sigma(\omega), \sigma(\omega')])$ is approximately $\varrho(\zeta^-\tilde{\omega}^+)\theta_{-L(v)}(\pi(\zeta^-\sigma^+(\tilde{\omega}^+)))\nu^+(\tilde{A}^+[\sigma(\omega), \sigma(\omega')])$ whenever ζ^+ is close enough to $\tilde\omega^+$. As $\omega'\to\omega$ the diameter of $\tilde A^+[\omega,\omega']$ tends to zero while $\tilde\omega^+\in\tilde A^+[\omega,\omega']$, so that $\frac{\nu^+(\sigma_+(\tilde A^+[\omega,\omega']))}{\nu^+(\tilde A^+[\omega,\omega'])}\to\frac{1}{g(\tilde\omega^+)}$. Therefore,

$$\ell(\omega^-\omega^+)=\frac{\int_{A^-}\varrho(\zeta^-\sigma^+(\tilde\omega^+))\theta_{-L(v)}(\pi(\zeta^-\sigma^+(\tilde\omega^+)))\,d\nu^-(\zeta^-)}{\int_{A^-}\varrho(\zeta^-\tilde\omega^+)\theta_{-v}(\pi(\zeta^-\tilde\omega^+))\,d\nu^-(\zeta^-)}\cdot\frac{1}{g(\tilde\omega^+)}$$

Letting $\tilde\eta^+$ be the future coding of $\pi(\omega)+v$ corresponding to η^+ we get

$$\ell(\omega^-\eta^+)=\frac{\int_{A^-}\varrho(\zeta^-\sigma^+(\tilde\eta^+))\theta_{-L(v)}(\pi(\zeta^-\sigma^+(\tilde\eta^+)))\,d\nu^-(\zeta^-)}{\int_{A^-}\varrho(\zeta^-\tilde\eta^+)\theta_{-v}(\pi(\zeta^-\tilde\eta^+))\,d\nu^-(\zeta^-)}\cdot\frac{1}{g(\tilde\eta^+)}.$$

Note that since $\omega^-\tilde\omega^+$ and $\omega^-\tilde\eta^+$ are two symbolic codings of a single point $x+v$ in $\mathrm{Int}(\pi([00]))$, $\sigma(\omega^-\tilde\omega^+)$ and $\sigma(\omega^-\tilde\eta^+)$ are two symbolic codings of the point $L(x+v)$ in $\mathrm{Int}(\pi([0])$. Hence, $\pi(p_+^{-1}(\sigma_+(\tilde\omega^+)))$ and $\pi(p_+^{-1}(\sigma_+(\tilde\eta^+)))$ are the same local stable manifold inside $\pi([0])$. Both points $\pi(\omega^-\sigma_+\tilde\omega^+)$ and $\pi(\omega^-\sigma_+\tilde\eta^+)$ lie on the intersection of this local stable manifold and the local unstable manifold $\pi(p_-^{-1}(\omega^-))$ inside $\pi([0])$, so they must coincide.

Repeating the argument at the end of Section 3.1 completes the proof. Since $x+v$ is not on the boundary of the partition $\bigvee_{0\le k<N}L^{-k}\mathcal{P}$, $\tilde\omega^+$ and $\tilde\eta^+$ agree on at least N symbols. Now Hölder continuity of ϱ and g implies that the ratio $\ell(\omega^-\omega^+)/\ell(\omega^-\eta^+)$ can be made arbitrarily close to one when by choosing N sufficiently large, so that ℓ is torus continuous.

So far, we have completed the proof that L is C^{1+H} in the new charts in the case that $A_0\cap L^{-1}A_0$ has non-empty interior. An essentially identical argument shows that if $A_0\cap L^{-n}A_0$ has non-empty interior, then L^n is C^{1+H} in the new charts. (The only modification is that the g-function has to be replaced by $g^{(n)}$ defined by $g^{(n)}(x)=g(x)g(\sigma(x))\cdots g(\sigma^{n-1}x)$). Since Anosov automorphisms are topologically mixing, $A_0\cap L^{-n}A_0$ has non-empty interior for all sufficiently large n. In particular there is n such that L^n and L^{n+1} are both C^{1+H} diffeomorphisms. It follows that $L=(L^n)^{-1}\circ L^{n+1}$ is C^{1+H} as required.

3.3. Cohomology of ϕ and the geometric potential of L in the new atlas.

Lemma 9. *Let L and $\mathcal{P}$ be as above. There exist $\gamma>0$ and $k>0$ such that if $R\subset A_0$ is of the form $R=\pi(C_-\times S)$ where C_- is an n-cylinder in Ω_- and $S\subset[0]\subset\Omega_+$, then $\mu(R)\le ke^{-\gamma n}\mu(\pi\circ p_+^{-1}S)$.*

The proof is an application of the product structure outlined in Section 2.2 together with the fact that ν^- is a g-measure with g_- bounded away from 1.

Lemma 10. *The map L is expanding in the unstable direction in the new coordinate system: for any finite sub-atlas there exists $n \in \mathbb{N}$ such that for any $x \in \mathbb{T}^2$ and for any charts in the sub-atlas containing x and $L^n x$ respectively, $D_u L^n > 2$ when computed in the respective charts.*

Proof. Let the finite sub-atlas be $\{\alpha_{u_1}, \ldots, \alpha_{u_N}\}$. Let M and M' be positive constants such that $\frac{1}{M} \le \theta_{u_i}(x) \le M$ for $1 \le i \le N$ and all of the maps $\alpha_{u_i} \circ \alpha_{u_j}^{-1}$ have derivatives between M' and M'^{-1} when $1 \le i, j \le N$. By compactness, there exists $\delta > 0$ such that for each $x \in \mathbb{T}^2$, there exists an i with $x + u_i \in A_0$ and $d(x + u_i, \partial A_0) > \delta$. Let n be a fixed integer sufficiently large that $\lambda^{-n} < \delta$ and also satisfying $e^{\gamma n} > 2kM^2 M'^2$, where λ is the expanding eigenvalue of L and k, γ are as in Lemma 9.

Let $u, v \in \{u_1, ..., u_N\}$ be such that $x + u$ and $L^n x + v$ both lie in $\mathrm{int}_\delta(A_0)$. Let $B_1 + u$ be a rectangle in A_0 whose projection in A_0 onto the stable direction is all of the stable manifold segment defining A_0 and whose unstable projection in A_0 is sufficiently narrow that $L^n B_1 + v \subset A_0$. Let B_2 be the rectangle in A_0 whose projection onto the stable direction is the stable manifold segment defining A_0 and whose unstable projection is the same as that of $L^n B_1 + v$. Then by Lemma 9,

$$\mu(L^n B_1 + v) \le k e^{-\gamma n} \mu(B_2). \tag{11}$$

We then have

$$\begin{aligned} \mu(L^n B_1 + v) &\ge \tfrac{1}{M}\mu(L^n B_1) \\ \mu(L^n B_1) &= \mu(B_1) \\ \mu(B_1) &\ge \tfrac{1}{M}\mu(B_1 + u). \end{aligned} \tag{12}$$

Combining equations (11) and (12), by the choice of n we see

$$\mu(B_2) \ge \frac{e^{\gamma n}}{kM^2}\mu(B_1 + u) \ge 2M'^2 \mu(B_1 + u).$$

Shrinking B_1 so that $B_1 + u$ shrinks to the segment of the stable manifold of x lying in A_0, we deduce the unstable derivative of L^n in the (α_u, α_v) charts is at least $2M'^2$. Now if u' and v' are such that $\alpha_{u'}$ and $\alpha_{v'}$ are arbitrary charts in the sub-atlas containing x and $L^n x$ in their domain then the unstable derivative of L^n in the $(\alpha_{u'}, \alpha_{v'})$ charts is at least 2. This completes the proof. □

Lemma 11. *There exists $M > 0$ such that for any $n \in \mathbb{N}$, any cylinder set C in Ω of the form $[0a_1 \ldots a_{n-1}0]$, and any $\omega \in C$ we have*

$$\frac{1}{M} \le |D_u L^n(\pi(\omega))| \cdot \exp(S_n \phi(\pi(\omega))) \le M,$$

where the unstable derivative of L is computed using the new charts.

Proof. The proof is based on a standard argument that the fibre maps of uniformly expanding maps have bounded distortion (see e.g. [21, Chapter III]). Suppose that x, y are points in A_0 which lie on the same local unstable manifold and are such that $x = \pi(\omega), y = \pi(\eta)$ with $\omega, \eta \in [0a_1 \dots a_{n-1}0] \subset \Omega$. Recall from Section 3.2 that in terms of charts of the new atlas, the map L has the form $\alpha_{v_1} \circ L \circ \alpha_{v_0}^{-1}(a, b) = (f_1(a), f_2(b))$ where the functions f_1 and f_2, which depend on the choice of v_0 and v_1, are differentiable with Hölder continuous derivatives. Since $\sigma^n(\omega), \sigma^n(\eta) \in [0]$ we have that both $L^n(x)$ and $L^n(y)$ are in A_0 and hence $d(L^j(x), L^j(y)) \leq \lambda^{-(n-j)}$ for $0 \leq j < n$, where λ is the expanding eigenvalue of L (and here the distance is computed using the original metric). Denote by $f_{1,j}$ the first component of L computed in the charts corresponding to $L^j x$ and $L^{j+1}x$. Applying the chain rule we see that

$$\left|\frac{D_u L^n(x)}{D_u L^n(y)}\right| = \prod_{j=0}^{n-1} \left|\frac{f'_{1,j}(L^j x)}{f'_{1,j}(L^j y)}\right|.$$

It follows from Hölder continuity of the derivatives and Lemma 10 that there are $K > 0$ and $\gamma \in (0, 1)$ such that for $0 \leq j < n$

$$\left|\frac{f_{1,j}(L^j(x))}{f_{1,j}(L^j(y))}\right| \leq 1 + Kd(L^j(x), L^j(y))^\gamma \leq 1 + K\lambda^{-(n-j)}.$$

Setting $M = \prod_{j=1}^{\infty}(1 + K\lambda^{-j})$ we obtain that $|D_u L^n(x)|/|D_u L^n(y)| \leq M$ for all x, y lying in a segment of the local unstable manifold contained in a single partition element.

Now suppose C is a cylinder set $[a_0 a_1 \dots a_n]$ in Ω with $a_0 = a_n = 0$. Let ω^- be a compatible past and set $U = \pi(\omega^- C)$, a piece of unstable manifold that is mapped bijectively by L^n onto a fibre of the unstable manifold crossing the partition element A_0. By the mean value theorem, the length (in the new charts) of $L^n U$ (which is the same as the width of the 0 partition element) is the product of the length of U and the unstable derivative at some point $u \in U$. Since coordinates (and hence lengths) in the unstable direction are computed using by μ-measures the measure $\mu = \pi_* \nu$, this gives, for any $\omega \in U$,

$$\frac{1}{M} \leq |(D_u L^n)(\pi(\omega))| \cdot \nu(C) \leq M.$$

Now applying the Bowen definition [2] for the Gibbs state ν of the potential $\phi \circ \pi$ together with the fact that $P_{\text{top}}(\sigma, \phi \circ \pi) = 0$,

$$\frac{1}{M'} \exp(S_n \phi(\pi(\omega))) \leq \mu(C) \leq M' \exp(S_n \phi(\pi(\omega)))$$

Substituting in the previous inequality gives the required statement. □

Lemma 12. *Let ϕ be as in the statement of Theorem 1, and let the charts be constructed as above. Then potential $\phi(x)$ is cohomologous to $-\log|D_uL(x)|$, where the unstable derivative is computed using the new charts.*

Proof. We rely on Livšic's theorem [13]: if T is a hyperbolic dynamical system and ψ is a Hölder continuous function such that $S_n\psi(p) = 0$ whenever $T^n p = p$, then ψ is a coboundary (with Hölder continuous transfer function).

As a corollary, if Ω is a mixing subshift of finite type and there exists an M such that $|S_{n+1}\psi(\omega)| \le M$ whenever $\omega \in [0]$ and $\sigma^n\omega \in [0]$, then ψ is a Hölder coboundary.

Lemma 11 shows that the function $\psi(\omega) = \chi \circ \pi(\omega)$ where $\chi(x) = \log|D_uL(x)| + \phi(x)$ satisfies the hypothesis of this corollary of Livšic's theorem, so that ψ is a Hölder coboundary. It follows that χ sums to zero around any periodic orbit in $\mathbb{T}^2$, so that χ is also a Hölder coboundary, using Livšic's theorem again. □

4. Application to the smooth conjugacy problem.

In this section we explicitly construct a countable family of Hölder potentials in the homotopy class of the toral automorphism L whose geometric potentials have identical pressure functions, yet they are not C^1 conjugate.

Lemma 13. *Let L be an automorphism of $\mathbb{T}^2$, let $k \in \mathbb{N}$ and let $M_k(x) = kx \bmod 1$. Then for any continuous function ϕ on $\mathbb{T}^2$*

$$P_{\mathrm{top}}(L, \phi) = P_{\mathrm{top}}(L, \phi \circ M_k).$$

Proof. We use the topological definition of pressure:

$$P_{\mathrm{top}}(L,\phi) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\sup\left\{\sum_{x\in E} e^{S_n\phi(x)} : E \text{ is } (n,\epsilon)\text{-separated}\right\},$$

where a subset E of $\mathbb{T}^2$ is (n,ϵ)-*separated* (with respect to L) if for any distinct elements $x, y \in E$, there exists $0 \le j < n$ such that $d(L^j x, L^j y) \ge \epsilon$.

Denote $\phi_k = \phi \circ M_k$. We first show that $P_{\mathrm{top}}(L, \phi_k) \ge P_{\mathrm{top}}(L, \phi)$. Let E be an (n, ϵ)-separated subset of $\mathbb{T}^2$. We define a subset E' of $\mathbb{T}^2$ by

$$E' = M_k^{-1}(E) = \{(x + \mathbf{n})/k : x \in E, \mathbf{n} \in \{0, \dots, k-1\}^2\}$$

and claim E' is $(n, \frac{\epsilon}{k})$-separated. In the case when $x \in E$ and $\mathbf{m}, \mathbf{n}$ are distinct elements of $\{0, \dots, k-1\}^2$, we claim $d(L^i(\frac{x+\mathbf{m}}{k}), L^i(\frac{x+\mathbf{n}}{k})) \geq \frac{1}{k}$ for each i. Since L is an automorphism, it suffices to show that $d(L^i(\frac{\mathbf{p}}{k}), 0) \geq \frac{1}{k}$ for each $\mathbf{p} \in \{0, \frac{1}{k}, \dots, \frac{k-1}{k}\}^2 \setminus \{(0,0)\}$ and $i \in \mathbb{N}$. Since the matrix A defining L has an inverse with integer entries, it is not hard to see that L is a permutation of the points $\{0, \dots, \frac{k-1}{k}\}^2$. Since L is injective, it follows that $d(L^i(\frac{\mathbf{p}}{k}), 0) \geq \frac{1}{k}$ for each i. In the case when x, y are distinct elements of E and $\mathbf{m}, \mathbf{n}$ are elements of $\{0, \dots, k-1\}^2$ (not necessarily distinct), letting $u = \frac{x+\mathbf{m}}{k}$ and $v = \frac{y+\mathbf{n}}{k}$, we have

$$d(L^i u, L^i v) \geq \tfrac{1}{k} d(M_k(L^i u), M_k(L^i v)) = \tfrac{1}{k} d(L^i x, L^i y).$$

Since $\max_{i<n} d(L^i x, L^i y) \geq \epsilon$, it follows that $\max_{i<n} d(L^i u, L^i v) \geq \frac{\epsilon}{k}$. Hence we have established that E' is $(n, \frac{\epsilon}{k})$-separated as required.

Note $S_n\phi_k(\frac{x+\mathbf{m}}{k}) = S_n\phi(x)$ for each $x \in E$ and $\mathbf{m} \in \{0, \dots, k-1\}^2$. Therefore

$$\sup\left\{\sum_{x\in E} e^{S_n\phi_k(x)} : E \text{ is } (n, \tfrac{\epsilon}{k})\text{-separated}\right\}$$
$$\geq k^2 \sup\left\{\sum_{x\in E} e^{S_n\phi(x)} : E \text{ is } (n, \epsilon)\text{-separated}\right\},$$

which gives $P_{\text{top}}(L, \phi_k) \geq P_{\text{top}}(L, \phi)$.

For the converse inequality, we first claim that any $u, v \in \mathbb{T}^2$ and for any positive $\epsilon < 1/(2k\|A\|)$, the following implication holds:

$$d(u,v) < \epsilon \text{ and } d(M_k(Lu), M_k(Lv)) < k\epsilon \text{ implies } d(Lu, Lv) < \epsilon. \tag{13}$$

Again, by the linearity of L, it suffices to show that if $d(u, 0) < \epsilon$ and $d(M_k(Lu), 0) < k\epsilon$ then $d(Lu, 0) < \epsilon$. To verify this claim, suppose $d(u, 0) < \epsilon$. By the choice of ϵ, $d(Lu, 0) < \frac{1}{2k}$ so that 0 is the closest element of $M_k^{-1}\{0\}$ to Lu. Since $d(M_k(Lu), 0) < k\epsilon$, the fact that M_k locally expands distances by a factor of k implies that $d(Lu, 0) < \epsilon$ as required.

Let $\epsilon < \frac{1}{2k\|A\|}$ and let E' be an (n, ϵ) separated set in $\mathbb{T}^2$. We define a relation R on E' by

$$uRv \quad \Leftrightarrow \quad \max_{0\leq i<n} d(L^i M_k(u), L^i M_k(v)) < \frac{\epsilon}{2k}.$$

Equivalently uRv iff $\max_{0\leq i<n} d(M_k(L^i u), M_k(L^i v)) < \frac{\epsilon}{2k}$, since $L \circ M_k = M_k \circ L$. We then take the transitive closure of R to form an equivalence relation $\sim$ on E'. That is, $u \sim v$ if there exist $u_0, u_1, \dots, u_l$ with $u_0 = u$, $u_l = v$ and $u_{i-1}Ru_i$ for $i = 1, \dots, l$. We claim that each $\sim$-equivalence class has at most k^2 elements. We prove this by contradiction. Suppose C is a $\sim$-equivalence class containing at least k^2+1 elements. We construct a subset D of cardinality exactly k^2+1 such that there is a path between any two elements of D using steps in R. To see this, fix an initial element of u_0 of C, enumerate the other elements of C and for each such element u, find an R-path, from the definition of $\sim$ connecting u_0 to u. We now build D by adding the elements of the paths one at a time until the cardinality is exactly k^2+1. (At each step when a vertex is to be included, D may either increase by one element if the vertex is new; or remain the same if the vertex has already been added.) By the construction, each element of D is connected by R to a previous element of D.

Let $D=\{u_0,\ldots,u_{k^2}\}$. By the triangle inequality and the definition of R, $d(M_k(L^i(u_0)),M_k(L^i(u_j)))<\frac{k\epsilon}{2}$ for each j (since we can get from u_0 to u_j along an R path of length at most k^2). In particular, $d(M_k(u_0),M_k(u_j))<\frac{k\epsilon}{2}$ for each j. Using the fact that M_k locally expands distances by a factor of k, for each $0\le j\le k^2$, u_j differs from u_0 by an element of $M_k^{-1}\{0\}=\{0,\frac{1}{k},\ldots,\frac{k-1}{k}\}^2$ plus a term of size at most $\frac{\epsilon}{2}$. By the pigeonhole principle, there exist $0\le j<j'\le k^2$ such that u_j and $u_{j'}$ differ by at most ϵ. Since $u_j\sim u_{j'}$, we see that $d(L^iM_k(u_j),L^iM_k(u_{j'}))<k\epsilon$ for $i=0,\ldots,n$. Applying (13) inductively we see $d(L^iu_j,L^iu_{j'})<\epsilon$ for $i=0,\ldots,n$. This contradicts the initial assumption that E' was (n,ϵ)-separated.

Hence we have shown that each $\sim$-equivalence class in E' has at most k^2 elements. Let the equivalence classes be $C_1,\ldots,C_M$; and for each equivalence class, pick $u_i\in C_i$ for which $S_n\phi_k(u_i)$ is maximal in the equivalence class. We now have

$$\sum_{u\in C_i}\exp(S_n\phi_k(u))\le k^2\exp(S_n\phi_k(u_i)).$$

Summing over the equivalence classes, we obtain

$$\sum_{u\in E'}\exp(S_n\phi_k(u))\le k^2\sum_{i=1}^{M}\exp(S_n\phi_k(u_i)).$$

Let $x_i=M_k(u_i)$ for each i. Since $S_n\phi_k(u_i)=S_n\phi(x_i)$, rearranging the above inequality gives

$$\sum_{i=1}^{M}\exp(S_n\phi(x_i))\ge\frac{1}{k^2}\sum_{u\in E'}\exp(S_n\phi_k(u)).$$

Finally, we claim that $\{x_1,\ldots,x_M\}$ is $(n,\frac{\epsilon}{2k})$-separated. If not then, there there exist j,l such that $d(L^ix_j,L^ix_l)<\frac{\epsilon}{2k}$ for $i=0,\ldots,n-1$.

Then, since $x_j = M_k(u_j)$ and $x_l = M_k(u_l)$, we see from the definition of R that $u_j R u_l$. This contradicts the assumption that the u_i's belong to distinct equivalence classes.

Hence we have shown

$$\sup\left\{\sum_{x\in E} e^{S_n\phi(x)} : E \text{ is } (n, \tfrac{\epsilon}{2k})\text{-separated}\right\} \geq \frac{1}{k^2}\sup\left\{\sum_{u\in E'} e^{S_n\phi_k(u)} : E \text{ is } (n,\epsilon)\text{-separated}\right\},$$

It follows that $P_{\mathrm{top}}(L,\phi) \geq P_{\mathrm{top}}(L,\phi_k)$ as required. □

Proof of Theorem 2. Let L be the Anosov automorphism of the torus given by the matrix $\begin{pmatrix}1&1\\1&0\end{pmatrix}$. Note that $(\frac{1}{2},0),(\frac{1}{2},\frac{1}{2}),(0,\frac{1}{2})$ is the unique period 3 orbit of L. Let ϕ be a Hölder continuous function of the torus of pressure 0 such that $\phi(0,0) \neq \frac{1}{3}(\phi(\frac{1}{2},0)+\phi(\frac{1}{2},\frac{1}{2})+\phi(0,\frac{1}{2}))$ and let $\phi_2(x)=\phi(2x)$ as above. Then

$$\tfrac{1}{3}(\phi_2(\tfrac{1}{2},0)+\phi_2(\tfrac{1}{2},\tfrac{1}{2})+\phi_2(0,\tfrac{1}{2})) = \phi(0,0) \neq \tfrac{1}{3}(\phi(\tfrac{1}{2},0)+\phi(\tfrac{1}{2},\tfrac{1}{2})+\phi(0,\tfrac{1}{2})).$$

We conclude the proof by showing that if T and T_2 are the area-preserving Anosov diffeomorphisms obtained from ϕ and ϕ_2 respectively as in Theorem 1, then T and T_2 are not conjugate, but they satisfy $P_{\mathrm{top}}(T,-sD_uT) = P_{\mathrm{top}}(T_2,-sD_uT_2)$ for all $s\in\mathbb{R}$.

Let h be the conjugacy between T and L obtained in the proof of Theorem 1. Similarly, let h_2 be the conjugacy between T_2 and L. The theorem guarantees that $-\log|D_uT|\circ h$ is cohomologous to ϕ and $-\log|D_uT_2|\circ h_2$ is cohomologous to ϕ_2. Let $p = h(\frac{1}{2},0)$ and notice that $\{p,Tp,T^2p\}$ is the unique period 3 orbit of T. Similarly let $p_2 = h_2(\frac{1}{2},0)$ so that $\{p_2,T_2p_2,T_2^2p_2\}$ is the unique period 3 orbit of T_2. Since $-\log|D_uT|\circ h$ is cohomologous to ϕ, we see that

$$|D_uT^3(p)| = |D_uT^3(Tp)| = |D_uT^3(T^2p)| = e^{\phi(\frac{1}{2},0)+\phi(\frac{1}{2},\frac{1}{2})+\phi(0,\frac{1}{2})},$$

while

$$|D_uT_2^3(p_2)| = e^{\phi_2(\frac{1}{2},0)+\phi_2(\frac{1}{2},\frac{1}{2})+\phi_2(0,\frac{1}{2})} = e^{3\phi(0,0)}.$$

Since differentiable conjugacies preserve unstable multipliers, we see that T and T_2 are not differentiably conjugate.

However,

$$\begin{aligned} P_{\text{top}}(T, -s\log|D_uT|) &= P_{\text{top}}(hLh^{-1}, -s\log|D_uT|) \\ &= P_{\text{top}}(L, -s\log|D_uT| \circ h) \\ &= P_{\text{top}}(L, -s\phi) \end{aligned}$$

and similarly $P_{\text{top}}(T_2, -s\log|D_uT_2|) = P_{\text{top}}(L, -s\phi_2)$. By Lemma 13, $P_{\text{top}}(L, -s\phi) = P_{\text{top}}(L, -s\phi \circ M_2) = P_{\text{top}}(L, -s\phi_2)$ for all $s \in \mathbb{R}$ so that $P_{\text{top}}(T, -s\log|D_uT|) = P_{\text{top}}(T_2, -s\log|D_uT_2|)$ for all s. □

References

[1] D. V. Anosov, *Geodesic flows on closed Riemannian manifolds*, Proc. Stek. Inst. **90** (1967)

[2] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Mathematics, Vol. 470. corr. 2nd rev. ed. (1st ed. 1975) Springer (2017).

[3] D. Capocaccia, *A definition of Gibbs states for a compact set with $\mathbb{Z}^\nu$-action*, Comm. Math. Phys. **48**, (1976), 85-88.

[4] E. Cawley, *The Teichmüller space of an Anosov diffeomorphism of $\mathbb{T}^2$*, Invent. Math. 112 (1993), no. 2, 351–376.

[5] J. DeWitt and A. Gogolev, *Dominated splitting from constant periodic data and global rigidity of Anosov automorphisms*, preprint arXiv:2309.07323

[6] J. Franks, *Anosov diffeomorphisms on tori*, Trans. AMS **145**, 117-125 (1969)

[7] A. Gogolev, *Smooth conjugacy of Anosov diffeomorphisms on higher dimensional tori*, Journal of Modern Dynamics, 2, no. 4, 645-700 (2008)

[8] A. Gogolev and F. Rodriguez Hertz, *Smooth rigidity for very non-algebraic Anosov diffeomorphisms of codimension one*, Israel Journal of Mathematics, to appear.

[9] N. Haydn, *On Gibbs and equilibrium states*, Ergodic Theory Dyn. Syst. **7**, 119-132 (1987).

[10] N. Haydn and D. Ruelle, *Equivalence of Gibbs and Equilibrium States for Homeomorphisms Satisfying Expansiveness and Specification*, Commun. Math. Phys. **148**, 155-167 (1992).

[11] A. Erchenko, *Flexibility of Lyapunov exponents with respect to two classes of measures on the torus*, Ergodic Theory and Dynamical Systems, **42**, Issue 2, 737 – 776 (2022).

[12] B. Kalinin, V. Sadovskaya, *On Anosov diffeomorphisms with asymptotically conformal periodic data*, Ergodic Theory and Dynamical Systems, **29** Issue 1, 117 - 136 (2009).

[13] A. N. Livšic, *Cohomology of dynamical systems*, Izv. Akad. Nauk SSSR Ser. Mat. **36**, 1296–1320 (1972).

[14] R. de la Llave, J. M. Marco, and R. Moriyón, *Canonical perturbation theory of Anosov systems and regularity results for the Livšic cohomology equation*, Ann. of Math. **(2) 123** no.3, 537–611 (1986).

[15] R. de la Llave, *Invariants for smooth conjugacy of hyperbolic dynamical systems II*, Commun. Math. Phys. **109**, 369-378 (1987)

[16] R. de la Llave and R. Moriyón, *Invariants for smooth conjugacy of hyperbolic dynamical systems IV*, Commun. Math. Phys. **116**, 185-192 (1988)
[17] R. de la Llave, *Smooth Conjugacy and S-R-B Measures for Uniformly and Non-Uniformly Hyperbolic Systems*, Commun. Math. Phys. **150**, 289-320 (1992)
[18] A. Manning, *There are no new Anosov diffeomorphisms on tori*, Amer. J. Math. **96**, 422-429 (1974).
[19] J. M. Marco and R. Moriyón, *Invariants for smooth conjugacy of hyperbolic dynamical systems I*, Commun. Math. Phys. **109**, 681-689 (1987)
[20] J. M. Marco and R. Moriyón, *Invariants for smooth conjugacy of hyperbolic dynamical systems IV*, Commun. Math. Phys. **112**, 317-333 (1987)
[21] R. Mañé, *Ergodic Theory and Differentiable Dynamics*, Springer (1987)
[22] S. Newhouse, *On codimension one Anosov diffeomorphisms*, Amer. J. Math. **92**, 761-770 (1970).
[23] M. Pollicott and H. Weiss, *Free Energy as a Dynamical Invariant (or Can You Hear the Shape of a Potential?)*, Commun. Math. Phys. **240**, 457–482 (2003)
[24] D. Ruelle, *Thermodynamic formalism. The mathematical structures of classical equilibrium statistical mechanics*, Addison-Wesley, 1978.
[25] S. Smale, *Differentiate dynamical systems*, Bull A.M.S. **73**, 747-817 (1967).
[26] P. Walters, *Ruelle's operator theorem and g-measures*, Trans. Amer. Math. Soc. **214** (1975), 375-387.

Department of Mathematics, The City College of New York, New York, NY, 10031, USA
Email address: tkucherenko@ccny.cuny.edu

Department of Mathematics and Statistics, University of Victoria, Victoria, BC Canada
Email address: aquas@uvic.ca