RADEMACHER BOUNDED FAMILIES OF OPERATORS ON L_1

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Abstract. We give a characterization of R-bounded families of operators on L_1. We then use this result to study sectorial operators on L_1. We show that if A is an R-sectorial operator on L_1 then, for any $\epsilon > 0$, there is an invertible operator $U : L_1 \to L_1$ with $\|U - I\| < \epsilon$ such that for some strictly positive Borel function w, $U(\mathcal{D}(A))$ contains the weighted L_1-space $L_1(w)$.

1. Introduction

Let us recall that a closed operator A on a Banach space X is called sectorial with sectoriality angle ω if

- The domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ are dense
- A is one-to-one
- The spectrum $\sigma(A)$ is contained in a closed sector $\Sigma_\omega = \{\zeta \in \mathbb{C} : |\arg \zeta| \le \omega\}$
- For any $\omega < \phi < \pi$ there is a constant C_ϕ such that the resolvent $R(\zeta, A)$ satisfies the estimate
$$\|\zeta R(\zeta, A)\| \le C_\phi, \ |\arg(\zeta)| \ge \phi.$$

Note that the definition does not require A to be invertible. If $\omega < \frac{\pi}{2}$ then the operator $-A$ generates a bounded analytic semigroup, $T_t = e^{-tA}$. Conversely if $-A$ is the generator of a bounded analytic semigroup then A is sectorial with $\omega < \pi/2$, provided it is one-one. For further discussion on sectorial operators see [2].

In applications involving L_p-maximal regularity of the abstract Cauchy problem or, more generally, the joint functional calculus of two commuting sectorial operators it is often important to know that a sectorial operator satisfies a stronger form of sectoriality, which we now introduce (see [8] and [11]).

We recall here that a collection of operators $\mathcal{T}$ on a Banach space X is called *R-bounded* if there is a constant C so that

$$(\mathbb{E}\|\sum_{j=1}^{n}\varepsilon_j T_j x_j\|^2)^{\frac12} \le C(\mathbb{E}\|\sum_{j=1}^{n}\varepsilon_j x_j\|^2)^{\frac12} \qquad x_1,\ldots,x_n \in X,\ T_1,\ldots,T_n \in \mathcal{T}.$$

Here $(\varepsilon_j)_{j=1}^\infty$ is a sequence of independent Rademacher functions. The Kahane-Khintchine inequality allows us to replace the exponent 2 in the definition by any $p \ge 1$. A is called *R-sectorial* with angle of R-sectoriality $\omega_R = \omega_R(A)$ if for every $\phi > \omega_R$ the collection of operators $\{\zeta R(\zeta, A) : \ |\arg \zeta| \ge \phi\}$ is R-bounded.

If, for example, A is invertible then R-sectoriality with $\omega_R(A) < \pi/2$ is necessary for L_p-maximal regularity of the abstract Cauchy problem $(1 < p < \infty)$; if further X is a UMD-space, it is also sufficient (see Weis [11] for details). Note that in a Hilbert space every sectorial operator is R-sectorial for the same angle.

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This note is concerned with the structure of R-sectorial operators on the Banach space $L_1 = L_1(K,\lambda)$ where K is a Polish space (i.e. a topological space which is homeomorphic to a separable complete metric space) and λ is a nonatomic σ-finite Borel measure. All such spaces are isometric to $L_1 = L_1[0,1]$, and so we will assume that K is a compact metric space and λ is a probability measure.

Our work is related to some previous results which suggest that it is rather restrictive for a sectorial operator A on L_1 to be R-sectorial. If A is a sectorial operator on L_1 which has H^∞-calculus (for some angle ω) then A is R-sectorial (for the same angle ω) [8]. We refer to [8] for the definition and discussion of the $H^\infty-$calculus. In [8] it was shown that if A has an $H^\infty-$calculus then A is bounded on any reflexive subspace of $\mathcal{D}(A)$ (with the graph norm); this had the implication that there are very few examples of sectorial operators with an $H^\infty-$calculus on L_1 and, in particular, essentially no reasonable differential operator can have this property. In [5] it was shown that there are no R-bounded strongly continuous semigroups on L_1 consisting of weakly compact operators; it also follows from the results of [5] that if A is an R-sectorial operator on L_1 then the resolvent $R(\zeta, A)$ can never be a weakly compact operator.

The simplest example of a sectorial operator on $L_1(K,\lambda)$ which has an $H^\infty-$calculus and hence is R-sectorial is the following. Given an a.e. positive function b we define the operator

$$Af(s) = b(s)f(s)$$

with domain

$$\mathcal{D}(A) = \left\{ f : \int |f(s)| b(s)^{-1} d\lambda(s) < \infty \right\}.$$

Note here that the domain is very large indeed; in fact for any $\epsilon > 0$ we can find a Borel set B with $\lambda(B) > 1-\epsilon$ and such that $L_1(B) \subset \mathcal{D}(A)$. Of course one can get further examples by considering $A' = UAU^{-1}$ for U any invertible operator with $\mathcal{D}(A') = U(\mathcal{D}(A))$.

In this note, we show that this example is typical. Precisely we show that if A is R-sectorial and $\epsilon > 0$ then there is an invertible operator $U : L_1 \to L_1$ with $\|U - I\| < \epsilon$ such that for some positive Borel function w we have $U(\mathcal{D}(A)) \supset L_1(w)$. This refines both the results of [5] and [8].

2. Operators on L_1

Let K be a compact metric space and suppose λ is a probability measure on K. We denote by $\mathcal{B}(K)$ the σ-algebra of Borel sets on K and by $\mathcal{M}(K)$ the space of Borel measures on K with the norm of total variation. We will utilize the so-called random measure representation of operators on L_1, developed in [6], [4] and [10].

A random measure on K is a map $s \to \mu_s$ from K into $\mathcal{M}(K)$ which is Borel for the weak*-topology on $\mathcal{M}(K)$. If the random measure satisfies the condition

$$\int_K |\mu_s|(B) d\lambda(s) \le C\lambda(B) \qquad B \in \mathcal{B}(K) \tag{2.1}$$

then it induces a bounded operator $T : L_1(\lambda) \to L_1(\lambda)$ given by the formula

$$Tf(s) = \int_K f(t) d\mu_s(t) \qquad \lambda - \text{a.e.} \tag{2.2}$$

and then $\|T\| \le C$.

Conversely every bounded linear operator $T : L_1(\lambda) \to L_1(\lambda)$ has an essentially unique random measure representation $s \to \mu_s^T$ and $\|T\|$ is the least constant C so that (2.1) holds for μ_s^T.

We may also associate to T a unique measure ρ_T on $K \times K$ given by

$$\rho_T(E) = \int_K \left(\int_K \chi_E(s,t) d\mu_s^T(t) \right) d\lambda(s) \qquad E \in \mathcal{B}(K \times K).$$

Thus

$$\rho_T(A \times B) = \int_A T\chi_B d\lambda.$$

The map $T \to \rho_T$ maps the space of all bounded operators on $L_1(K)$, denoted by $\mathcal{L}(L_1)$, onto an order-ideal in $\mathcal{M}(K \times K)$ consisting of all measures ρ such that

$$|\rho|(A \times B) \leq C\lambda(B), \qquad A, B \in \mathcal{B}(K \times K).$$

The space $\mathcal{L}(L_1(K,\lambda))$ is a complex Banach lattice and it is easily checked that if $T \in \mathcal{L}(L_1)$ then $\mu_s^{|T|} = |\mu_s^T|$ (λ-a.e.) and that $\rho_{|T|} = |\rho_T|$. Since it is a Banach lattice we can define as usual, using the Krivine calculus, an operator $(\sum_{j=1}^n |T_j|^2)^{\frac{1}{2}}$ for any $T_1, \ldots, T_n \in \mathcal{L}(L_1)$ (a full description of this construction is given in [9]).

The following result is implicitly contained in ideas of [6], and more explicitly in [7] :

Proposition 2.1. *Let $T_n : L_1 \to L_1$ be a uniformly bounded sequence of operators such that $\lim_{n\to\infty} \|\rho_{T_n}\| = 0$. Then given any $\epsilon > 0$ there is a Borel subset B of K with $\lambda(B) > 1 - \epsilon$ and $n \in \mathbb{N}$ so that we have*

$$\|T_n f\| \leq \epsilon \|f\| \qquad f \in L_1(B).$$

Proof. Let $\sigma_n = |\rho_{T_n}|$. Consider the measure ν_n on K given, for A Borel, by

$$\nu_n(A) = \sigma_n(A \times K) = \||T_n|\chi_A\|.$$

Then ν_n is absolutely continuous with respect to λ. Let w_n be its Radon-Nikodym derivative. Then, by our hypothesis,

$$\int w_n d\lambda = \sigma_n(K \times K) \to 0.$$

Therefore, $w_n \longrightarrow 0$ in measure. Hence there exists $n \in \mathbb{N}$ and B with $\lambda(B) > 1 - \epsilon$ so that $|w_n| < \epsilon$ on B.

If $f \in L_1(B)$ we have

$$\|T_n f\| \leq \int_{K \times K} |f(s)|\, d\sigma_n(s,t) = \int_B |f(s)| w_n(s)\, d\lambda(s) \leq \epsilon \|f\|.$$

∎

If $T \in \mathcal{L}(L_1)$ then we can write μ_s as given in (2.2) in the form

$$\mu_s = a(s)\delta_s + \mu_s' \qquad \lambda - \text{a.e.}$$

where $\mu_s'\{s\} = 0$ $\lambda-$a.e. and a is a bounded Borel function. (See for example [6]). Thus

$$Tf(s) = a(s)f(s) + \int_K f(t) d\mu_s'(t) \qquad \lambda - \text{a.e.}$$

If we define the diagonal part of T by

$$\Pi(T)f = a(s)f(s)$$

then $\rho_{\Pi(T)}$ is the restriction of ρ_T to the diagonal subset $\Delta = \{(s,s) : \ s \in K\}$. Thus

$$\rho_{\Pi(T)}(B) = \rho_T(B \cap \Delta).$$

Theorem 2.2. *Let $\mathcal{T}$ be a family of operators in $\mathcal{L}(L_1(K,\lambda))$. Then the following are equivalent:*

(i) $\mathcal{T}$ *is R−bounded.*
(ii) $\{(\sum_{k=1}^n a_k^2|T_k|^2)^{\frac{1}{2}} : \sum_{k=1}^n |a_k|^2 \le 1,\, T_1,...,T_n \in \mathcal{T},\, n \in \mathbb{N}\}$ *is uniformly bounded.*

Proof. Assume $\mathcal{T}$ is R-bounded, with

$$\mathbb{E}\|\sum_{k=1}^n \epsilon_k T_k x_k\| \le C\mathbb{E}\|\sum_{k=1}^n \epsilon_k x_k\|$$

for any $T_1, \dots, T_n \in \mathcal{T}$ and $x_1, \dots, x_n \in X$. Suppose $T_1, \dots, T_n \in \mathcal{T}$ and $a_1, \dots, a_n \in \mathbb{C}$ are such that $\sum_{k=1}^n |a_k|^2 \le 1$. Then, by Khintchine's inequality for lattices,

$$\left\|(\sum_{k=1}^n |a_k|^2|T_k|^2)^{\frac{1}{2}}\right\| \le M\mathbb{E}\left\||\sum_{k=1}^n \epsilon_k a_k T_k|\right\|$$

where M is an absolute constant. Choose any sequence of partitions $\mathcal{A}_m = (A_{mj})_{j=1}^{N_m}$ of K so that each $\mathcal{A}_{m+1}$ refines $\mathcal{A}_m$ and

$$\lim_{m\to\infty} \sup_{1\le j\le N_m} \operatorname{diam} A_{mj} = 0.$$

Then for any positive function $f \in L_1(K, \lambda)$ and any $T \in \mathcal{L}(L_1(K, \lambda))$ we have

$$|T|f = \lim_{m\to\infty} \sum_{j=1}^{N_m} |T(f\chi_{A_{mj}})| \qquad \lambda - \text{a.e.}$$

Thus, replacing T by $\sum_{k=1}^n \epsilon_k a_k T_k$ in the previous line yields

$$|\sum_{k=1}^n \epsilon_k a_k T_k|f = \lim_{m\to\infty} \sum_{j=1}^{N_m} |\sum_{k=1}^n \epsilon_k a_k T_k(f\chi_{A_{mj}})| \qquad \lambda - \text{a.e.}$$

Now, by R-boundedness

$$\begin{aligned}\mathbb{E}\int_K \sum_{j=1}^{N_m} |\sum_{k=1}^n \epsilon_k a_k T_k(f\chi_{A_{mj}})|d\lambda &= \sum_{j=1}^{N_m} \mathbb{E}\|\sum_{k=1}^n \epsilon_k a_k T_k(f\chi_{A_{mj}})\| \\ &\le C\sum_{j=1}^{N_m} \mathbb{E}\|\sum_{k=1}^n \epsilon_k a_k f\chi_{A_{mj}}\| \\ &= C\sum_{j=1}^{N_m} \|f\chi_{A_{mj}}\|\mathbb{E}|\sum_{k=1}^n \epsilon_k a_k| \\ &= C(\sum_{k=1}^n |a_k|^2)^{\frac{1}{2}} \sum_{j=1}^{N_m} \|f\chi_{A_{mj}}\| \\ &\le C\|f\|_{L_1}.\end{aligned}$$

It follows from Fatou's Lemma that

$$\mathbb{E}\left\| |\sum_{k=1}^n \epsilon_k a_k T_k| \right\| \le C$$

and hence

$$\|(\sum_{k=1}^n |a_k|^2|T_k|^2)^{\frac{1}{2}}\| \le CM.$$

We now prove that (*ii*) implies (*i*). First suppose $f \in L_1(K,\lambda)$ is positive and $T_1,\ldots,T_n \in \mathcal{L}(L_1(K,\lambda))$. Then if $a_1,\ldots,a_n \ge 0$ and $a_1^2+\cdots+a_n^2 = 1$ we have

$$\sum_{k=1}^n a_k|T_k|f \le (\sum_{k=1}^n |T_k|^2)^{\frac12} f.$$

The least upper bound of the left hand-side over all choices of $a_1,\ldots,a_n$ is $(\sum_{k=1}^n(|T_k|f)^2)^{\frac12}$ and so

$$(\sum_{k=1}^n(|T_k|f)^2)^{\frac12} \le (\sum_{k=1}^n |T_k|^2)^{\frac12} f.$$

Let us suppose C is a constant so that

$$\|(\sum_{k=1}^n |a_k|^2|T_k|^2)^{\frac12}\| \le C \qquad T_1,\ldots,T_n \in \mathcal{T},\ |a_1|^2+\cdots+|a_n|^2 = 1.$$

Suppose $f \in L_1$ and $T_1,\ldots,T_n \in \mathcal{T}$. Then

$$\begin{aligned}\mathbb{E}\|\sum_{k=1}^n \epsilon_k a_k T_k f\| &\le \|(\sum_{k=1}^n |a_k|^2|T_kf|^2)^{\frac12}\| \\ &\le \|(\sum_{k=1}^n |a_k|^2(|T_k||f|)^2)^{\frac12}\| \\ &\le \|(\sum_{k=1}^n |a_k|^2|T_k|^2)^{\frac12}|f|\ \| \\ &\le C\|f\|.\end{aligned}$$

In this situation, Theorem 2.2 of [5] implies that $\mathcal{T}$ is R-bounded. □

Proposition 2.3. *Suppose $\mathcal{T}$ is an R-bounded family of operators on $L_1(K,\lambda)$. Then the family of measures $\{\rho_T : T \in \mathcal{T}\}$ is relatively weakly compact in $\mathcal{M}(K\times K)$.*

Proof. Let

$$C = \sup\{\|(\sum_{j=1}^m |a_j|^2|T_j|^2)^{\frac12}\| : T_1,\ldots,T_m \in \mathcal{T},\ \sum_{j=1}^m |a_j|^2 \le 1,\ m\in\mathbb{N}\}$$

which is finite by Theorem 2.2. Now, if $T_1,\ldots,T_n \in \mathcal{T}$ then

$$\|\max_{1\le k\le n}|T_k|\| \le \|(\sum_{k=1}^n |T_k|^2)^{\frac12}\| \le Cn^{\frac12}.$$

The maximum here is computed in the lattice $\mathcal{L}(L_1)$.

Hence

$$\|\max_{1\le k\le n}|\rho_{T_k}|\ \|_{\mathcal{M}(K\times K)} \le Cn^{\frac12}.$$

Assume the set $\{\rho_T : T \in \mathcal{T}\}$ is not relatively weakly compact. Then there is a $\delta > 0$, a sequence $(T_k)_{k=1}^n$ and a sequence of disjoint open sets U_k in $K\times K$ so that $\rho_{T_k}(U_k) \ge \delta$ for all k (see e.g. [3]). Then

$$\|\max_{1\le k\le n}|\rho_{T_k}|\,\|_{\mathcal{M}(K\times K)} \ge \sum_{k=1}^n \rho_{T_k}(U_k) \ge \delta n \qquad n = 1,2,\ldots$$

which gives a contradiction. □

3. Applications to sectorial operators

In this sections, we give some applications of the above results to sectorial operators.

Proposition 3.1. *If A is R-sectorial and $\omega_R(A) < \pi/2$ then $\{e^{-tA} : 0 < t < \infty\}$ is an R-bounded semigroup. Conversely if A is sectorial and $-A$ generates an R-bounded semigroup then A is R-sectorial with $\omega_R(A) \le \pi/2$.*

If $-A$ is a sectorial operator which generates a semigroup $\{e^{-tA} : 0 < t < \infty\}$ with the property that $\{e^{-tA} : 0 < t \le 1\}$ is R-bounded then for any $\phi > \pi/2$ there exists M so that the set $\{\zeta R(\zeta, A) : \ |\arg(\zeta + M)| \ge \phi\}$ is R-bounded.

Proof. Our proof depends mainly on the two formulas

$$\zeta R(\zeta, A) = \int_0^\infty \zeta e^{\zeta t} e^{-tA} dt$$

and

$$e^{-tA} - (1 + tA)^{-1} = -\frac{1}{2\pi i} \int_{\Gamma_\nu} (e^{-t\zeta} - (1 + t\zeta)^{-1}) R(\zeta, A) d\zeta$$

where Γ_ν is a contour of the form $\{|s|e^{i(\text{sgn } s)\nu} : -\infty < s < \infty\}$ for any ν with $\nu > \omega(A)$.

Assuming that $\{e^{-At} : 0 < t < \infty\}$ is R-bounded we fix some angle $\frac{\pi}{2} < \varphi < \pi$. Then for any choice of numbers $\zeta_j = r_j e^{i\varphi_j}$ with $\varphi_j \ge \varphi$, $j = 1, ..., n$ we obtain

$$\begin{aligned}
\mathbb{E}\|\sum_{j=1}^n \epsilon_j \zeta_j R(\zeta_j, A) x_j\| &= \mathbb{E}\|\sum_{j=1}^n \int_0^\infty \epsilon_j r_j e^{i\varphi_j} e^{t r_j e^{i\varphi_j}} e^{-tA} x_j dt\| \\
&= \mathbb{E}\|\sum_{j=1}^n \int_0^\infty \epsilon_j e^{i\varphi_j} e^{s e^{i\varphi_j}} e^{-s/r_j A} x_j ds\| \\
&\le \int_0^\infty \mathbb{E}\|\sum_{j=1}^n \epsilon_j e^{i\varphi_j} e^{s e^{i\varphi_j}} e^{-s/r_j A} x_j\| ds \\
&\le C \int_0^\infty \max_j |e^{s e^{i\varphi_j}}| ds \cdot \mathbb{E}\|\sum_{j=1}^n \epsilon_j x_j\| \\
&\le C \int_0^\infty e^{s \cos\varphi} ds \cdot \mathbb{E}\|\sum_{j=1}^n \epsilon_j x_j\| \\
&\le \frac{C}{|\cos\varphi|} \cdot \mathbb{E}\|\sum_{j=1}^n \epsilon_j x_j\|.
\end{aligned}$$

Therefore A is R-sectorial with sectoriality angle $\omega_R(A) \le \pi/2$. Similarly, it follows that if A is R-sectorial and $\omega_R(A) < \pi/2$ then $\{e^{-tA} : 0 < t < \infty\}$ is R-bounded.

For the last statement suppose that C is a constant such that

$$(\mathbb{E}\|\sum_{j=1}^n \varepsilon_j e^{-t_j A} x_j\|^2)^{\frac{1}{2}} \le C(\mathbb{E}\|\sum_{j=1}^n \varepsilon_j x_j\|^2)^{\frac{1}{2}}$$

whenever $x_1, \dots, x_n \in X,\ 0 \le t_1, \dots, t_n \le 1$.

Then if $m \in \mathbb{N}$,

$$(\mathbb{E}\|\sum_{j=1}^n \varepsilon_j e^{-(m+t_j)A} x_j\|^2)^{\frac{1}{2}} \le CK^m(\mathbb{E}\|\sum_{j=1}^n \varepsilon_j x_j\|^2)^{\frac{1}{2}}$$

where $K = \|e^{-A}\|$. Now we show that the set $\{e^{-ut}e^{-tA} : 0 < t < \infty\}$ is R-bounded as long as $e^u > K$. For $x_1, ..., x_n \in X$ and $0 < t_1, \dots, t_n < \infty$ we obtain

$$(\mathbb{E}\|\sum_{j=1}^{n} \varepsilon_j e^{-t_j u} e^{-t_j A} x_j\|^2)^{\frac{1}{2}} = (\mathbb{E}\|\sum_{m=0}^{\infty} \sum_{m \le t_j < m+1} \varepsilon_j e^{-t_j u} e^{-t_j A} x_j\|^2)^{\frac{1}{2}}$$

$$\le C \sum_{m=0}^{\infty} K^m e^{-um} (\mathbb{E}\| \sum_{m \le t_j < m+1} \varepsilon_j e^{-u\tilde{t}_j} x_j\|^2)^{\frac{1}{2}}$$

where $0 \le \tilde{t}_j \le 1$. By the contraction principle

$$(\mathbb{E}\| \sum_{m \le t_j < m+1} \varepsilon_j e^{-u\tilde{t}_j} x_j\|^2)^{\frac{1}{2}} \le \max_{1 \le j \le n} |e^{-u\tilde{t}_j}| (\mathbb{E}\|\sum_{j=1}^{n} \varepsilon_j x_j\|^2)^{\frac{1}{2}} \le (\mathbb{E}\|\sum_{j=1}^{n} \varepsilon_j x_j\|^2)^{\frac{1}{2}}$$

Since $\sum_{m=0}^{\infty} K^m e^{-um}$ is finite for $u > \ln K$ we obtain the claim. Consequently, the set $\{\xi R(\xi, u + A) : |\arg \xi| > \phi\}$ is R-bounded.

Now for $\zeta \in \mathbb{C}$ with $|\arg(\zeta + M)| > \phi$, $M > u$ we can rewrite using $\xi - u = \zeta$,

$$\zeta R(\zeta, A) = (\xi - u) R(\xi - u) = (\xi - u) R(\xi, A + u) = \frac{\xi - u}{\xi} \xi R(\xi, A + u)$$

Since $|\frac{\xi - u}{\xi}| \le \frac{M}{M-u}$ the result follows quickly. □

It follows from results of [5] that if $-A$ is the generator of an semigroup such that e^{-tA} is weakly compact for $t > 0$, or if the resolvents $R(z, A)$ are weakly compact operators then A cannot be R-sectorial. The next Theorem strengthens this conclusion.

Theorem 3.2. *Suppose A is a sectorial operator on $L_1(K, \lambda)$. Assume that either:*
(i) A is R-sectorial for some angle ω, or
(ii) $-A$ is generator of a bounded semigroup such that $\{e^{-tA} : 0 < t \le 1\}$ is R-bounded.
Then there is a bounded function $a(\zeta, s)$ defined for $s \in K$ and $|\arg \zeta| > \omega$ such that

- *For each $s \in K$ the map $\zeta \to a(\zeta, s)$ is analytic.*
- *For each ζ the map $s \to a(\zeta, s)$ is Borel.*
- *$\lambda\{s : a(\zeta, s) = 0\} = 0$ for almost every ζ.*
- $$(R(\zeta, A)f)(s) = a(\zeta, s) f(s) + \int_K f(t) d\mu_s^\zeta(t) \qquad f \in L_1$$

  where $\mu_s^\zeta\{s\} = 0$.

Proof. We begin with the observation that, under either hypothesis, there exists $\phi < \pi$ and $M < \infty$ such that the set of operators $\{\zeta R(\zeta, A) : |\arg \zeta| \ge \phi, |\zeta| \ge M\}$ is R-bounded. Hence the set of measures $\{\rho_{R(\zeta, A)} : |\arg \zeta| \ge \phi, |\zeta| \ge M\}$ is relatively weakly compact.

Consider the map $\zeta \to \Pi(R(\zeta, A))$ which is an analytic map from the set $\mathcal{S} = \{\zeta : |\arg \zeta| > \omega\}$ into $\mathcal{L}(L_1)$. This induces an analytic map $F : \mathcal{S} \to L_\infty(K, \lambda)$ given by

$$\Pi(R(\zeta, A))f = F(\zeta)f.$$

Let us show that we can choose representatives so that $F(\zeta)(s) = a(\zeta, s)$ where a satisfies the first two conditions of the statement. Indeed let $\mathbb{D}$ be the unit disk and let $\varphi : \mathbb{D} \to \mathcal{S}$ be a conformal equivalence. Then $F \circ \phi$ can be expanded in a Taylor series around the origin and we may pick uniformly bounded Borel representatives b_n for the coefficients in the expansion so that

$$F(\varphi(z))(s) = \sum_{n=0}^{\infty} b_n(s) z^n \qquad \lambda-\text{a.e.},\ z \in \mathbb{D}.$$

Let

$$a(\zeta, s) = \sum_{n=0}^{\infty} b_n(s)(\varphi^{-1}(\zeta))^n.$$

Assume that the third condition fails. Then by Fubini's theorem there is a subset B of K with $\lambda(B) > 0$ so that for each $s \in B$ the set $\{\zeta : \ a(\zeta, s) = 0\}$ has positive planar measure. By analyticity, this implies $a(\zeta, s) \equiv 0$ for $s \in B$.

However $\rho_{n(n+A)^{-1}}$ converges weakly to ρ_I and hence so does $\rho_{\Pi(n(n+A)^{-1})}$. Thus $-na(-n, s)$ is weakly convergent to the constant function $1 \in L_1(K, \lambda)$. This is a contradiction. □

The next theorem shows that if a sectorial operator generates an R-bounded semigroup on L_1 then it is very similar to a bounded operator in the sense that its domain is sufficiently large to contain generic L_1-functions.

Theorem 3.3. *Let A be a sectorial operator on $L_1(K, \lambda)$ and assume that for some $\phi < \pi$ and $M < \infty$ the set $\{\zeta R(\zeta, A) : \ |\arg \zeta| \geq \phi,\ |\zeta| \geq M\}$ is R-bounded. Then for any $\epsilon > 0$ there is an invertible operator $U : L_1 \to L_1$ with $\|U - I\| < \epsilon$ and a density function $w > 0$ a.e. such that $L_1(w) \subset U^{-1}(\mathcal{D}(A))$.*

In particular, there is a closed subspace Y of $\mathcal{D}(A)$ isomorphic to L_1 so that $A : Y \to A(Y)$ is bounded (and thus Y is also closed in L_1).

Proof. According to Proposition 2.3 the set of measures $\rho_{\zeta R(\zeta,A)}$ for $|\arg \zeta| \geq \phi,\ |\zeta| \geq M$ is relatively weakly compact in $\mathcal{M}(K \times K)$. The sequence $(m(m + A)^{-1})_{m \geq M}$ converges in the strong operator topology to the identity. Therefore, $\rho_{m(m+A)^{-1}}$ converges weak* to ρ_I in $\mathcal{M}(K \times K)$ and hence converges weakly to ρ_I by weak compactness.

Fix $\epsilon > 0$. We may find a sequence of convex combinations $(T_n)_{n=1}^{\infty}$ of $\{m(m+A)^{-1}\}_{m=1}^{\infty}$ such that ρ_{T_n} converges to ρ_I in norm. Applying Proposition 2.1 to $(T_n - I)_{n=1}^{\infty}$ gives a sequence of Borel sets $E_n \subset K$ such that $\lambda(E_n) > 1 - 2^{-n}\epsilon$ and

$$\|T_n f - f\| \leq 2^{-n}\epsilon\|f\| \qquad f \in L_1(E_n).$$

Let us put $F_1 = E_1$ and then $F_n = E_n \setminus E_{n-1}$ for $n \geq 2$. We define $U : L_1 \to L_1$ by

$$Uf = \sum_{n=1}^{\infty} T_n(f\chi_{F_n}).$$

Thus $\|U - I\| \leq \epsilon$. Observe that $T_n : L_1 \to \mathcal{D}(A)$ and so AT_n is a bounded operator on L_1.

Define

$$w = \sum_{n=1}^{\infty} \|AT_n\|\chi_{F_n}$$

and assume $f \in L_1(w)$. Then

$$\|AU(f\chi_{F_n})\| = \|AT_n(f\chi_{F_n})\| \leq \int_{F_n} |f| w\, dt.$$

Hence $\sum_{k=1}^{\infty} AU(f\chi_{F_k})$ converges and, since A is closed, $Uf \in \mathcal{D}(A)$.

The last part of the theorem is deduced by fixing any n and note that if $Y = U(L_1(E_n))$ then A is bounded on Y and hence Y is closed both $\mathcal{D}(A)$ and X and is isomorphic to L_1 in both. □

Many differential operators on bounded domains have compact resolvents. Therefore we can use the results of [5] to show that they cannot be R-sectorial. In contrast, resolvents of differential operators on unbounded domains are, in general, not compact. An important example is the Laplacian Δ on $L_1(\mathbb{R}^n)$. Our corollary addresses this situation.

Corollary 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with locally Lipschitz boundary or $\Omega = \mathbb{R}^n$. Suppose that $A : \mathcal{D}(A) \subset L_1(\Omega) \longrightarrow L_1(\Omega)$ is a sectorial operator such that $\mathcal{D}(A)$ is contained in a Sobolev space $H_1^s(\Omega)$ for some $s > 0$. Then A does not generate an R-bounded semigroup.*

Proof. Assume the contrary, i.e. A generates an R-bounded semigroup. Then by Sobolev's embedding theorem [1] we have a continuous inclusion $H_1^s(\Omega) \hookrightarrow L_p(\Omega) \cap L_1(\Omega)$ for some $p > 1$. By Theorem 3.3 there is a closed subspace Y of $\mathcal{D}(A)$ on which A is bounded and so that Y is isomorphic to L_1. This implies that there is a subspace of $L_1(\Omega) \cap L_p(\Omega)$ which is isomorphic to L_1. If Ω is bounded this is an immediate contradiction since $L_1(\Omega) \cap L_p(\Omega) = L_p(\Omega)$ is reflexive. However even if Ω is unbounded this is still impossible. If $\Omega = \mathbb{R}^n$ we consider an isomorphism $J : L_1 \to L_1(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$. Then $J : L_1 \to L_p(\mathbb{R}^n)$ is a Dunford-Pettis operator and so if (f_n) is any normalized weakly null sequence in L_1 we have $\|Jf_n\|_p \to 0$. By passing to a subsequence we can assume $Jf_n \to 0$ a.e. But then (Jf_n) is also weakly null in $L_1(\mathbb{R}^n)$ and so $\|Jf_n\|_1 \to 0$. This gives a contradiction. □

This corollary is actually true for any set Ω for which Sobolev's embedding theorem holds. Sufficient geometrical properties of Ω for this to happen are discussed in [1].

References

[1] R. A. Adams, *Sobolev spaces*, Pure and Applied Mathematics, Vol. 65, Academic Press, New York-London, 1975.

[2] M. Cowling, I. Doust, A. McIntosh, and A. Yagi, *Banach space operators with a bounded H^∞ functional calculus*, J. Austral. Math. Soc. Ser. A **60** (1996), 51–89.

[3] J. Diestel, *Sequences and series in Banach spaces*, Graduate Texts in Mathematics, vol. 92, Springer-Verlag, New York, 1984.

[4] H. Fakhoury, *Représentations d'opérateurs à valeurs dans $L^1(X, \Sigma, \mu)$*, Math. Ann. **240** (1979), 203–212. (French)

[5] M. Hoffmann, N. Kalton, and T. Kucherenko, *R-bounded approximating sequences and applications to semigroups*, J. Math. Anal. Appl. **294** (2004), 373–386.

[6] N. J. Kalton, *The endomorphisms of L_p $(0 \le p \le 1)$*, Indiana Univ. Math. J. **27** (1978), 353–381.

[7] ———, *Linear operators on L_p for $0 < p < 1$*, Trans. Amer. Math. Soc. **259** (1980), 319–355.

[8] N. J. Kalton and L. Weis, *The H^∞-calculus and sums of closed operators*, Math. Ann. **321** (2001), 319–345.

[9] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces, II, Function spaces*, vol. 97, Springer-Verlag, Berlin, 1979.

[10] L. Weis, *On the representation of order continuous operators by random measures*, Trans. Amer. Math. Soc. **285** (1984), 535–563.

[11] ———, *Operator-valued Fourier multiplier theorems and maximal L_p-regularity*, Math. Ann. **319** (2001), 735–758.

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