COMMENTS ON THE PAPER: N. J. KALTON AND T. KUCHERENKO, OPERATORS WITH AN ABSOLUTE FUNCTIONAL CALCULUS, MATH. ANN. 346 (2010), NO. 2, 259–306.

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Introduction. Nigel Kalton was my Ph.D. thesis advisor at the University of Missouri-Columbia, where I arrived in 2001. My prior studies took place at Kharkov National University, and they focused broadly on functional analysis. I felt privileged to have the opportunity of working with Nigel, who was a world leading expert in this area.

I became involved with the topic of functional calculus for sectorial operators when I participated in the TULKA Internet Seminar[1] during the 2001/2002 academic year. This is an international workshop held annually and conducted online, which is geared towards for graduate students and postdoctoral fellows. The final event of this workshop is a live one week meeting of all participants over the Summer. This meeting is organized by the host institution in Europe, and in 2002 it took place in Blaubeuren, Germany. The philosophy of the workshop is to introduce students to a new topic and guide them along a series of lecture notes and assignments to an active research role. The Internet Seminar has been running each year since 1997, and it has covered a variety of topics related to evolution equations.

The selected theme of the 2001/2002 seminar was "Functional Calculus and Differential Operators", which was sparked by the recent breakthrough of Kalton, Lancien, and Weis on the problem of maximal regularity. I was excited about this material, and my involvement with the internet seminar resulted in several of my research projects. The direction of my graduate studies including the thesis were heavily shaped by the outcome of that academic year. The paper "Operators with an absolute functional calculus" [15] together with the survey article "Sectorial operators and interpolation theory" [16] is the culmination of my work on the subject.

[1]organized by the universities of **T**übingen, **UL**m, **KA**rlsruhe; see description of historical and current seminars at http://www.math.kit.edu/iana3/page/isem/en

Development of the paper. One of Nigel Kalton's remarkable strengths was his ability to apply deep insights and methods from classical Banach space theory to other areas. The paper under review here is an example of a journey that Nigel took into the world of partial differential equations and how far he was able to go. Already, his starting point was an achievement that concluded a forty year discussion on the outstanding L_p-maximal regularity problem.

The maximal regularity problem concerns solutions of the abstract Cauchy problem

$$\begin{cases} \dot{u}(t) + Bu(t) = h(t) & \text{for } t \in [0,T],\ 0 < T \leq \infty \\ u(0) = 0 \end{cases} \tag{1}$$

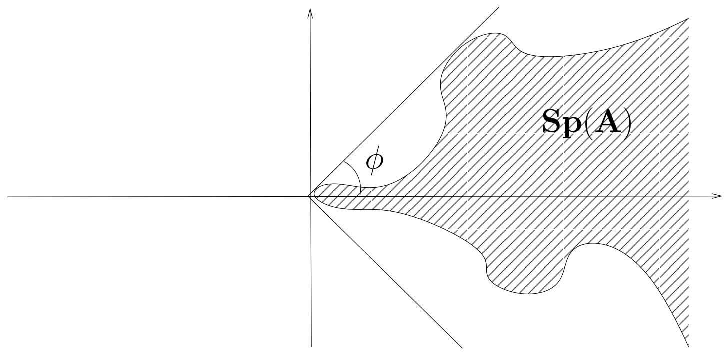
where B is a closed and densely defined operator on a Banach space X, and $h : \mathbb{R}_+ \longrightarrow X$ is a locally integrable function.

For $h \in L_p(X)$ $(1 < p < \infty)$ it is a straightforward argument to see that the solution $u \in L_p(X)$. We say that B has L_p-maximal regularity if it also follows that $\dot{u} \in L_p(X)$. Although this definition apparently depends on the value of p, it was Dore who showed in 1993 that in fact L_p-maximal regularity for some p implies the same for every p where $1 < p < \infty$ [11].

The cradle of the maximal regularity problem was a result published by DeSimon in 1964 [10]. He established that if X is a Hilbert space then every sectorial operator B (defined below) with angle of sectoriality less than $\pi/2$ has maximal regularity. A direct generalization of this theorem from Hilbert to Banach spaces fails, and many counter examples are known (see [8], [19]). However, the maximal regularity property appeared to hold for all concrete examples arising from partial differential equations on classical L_p spaces. Therefore, it was conjectured by Brézis in 1980 that the result of DeSimon could be extended to the case when $X = L_p$ for $1 < p < \infty$.

In 1999, Kalton and Lancien used Banach space techniques to show that Hilbert spaces among Banach spaces with unconditional bases are the only ones for which DeSimon's result holds [17]. Therefore, Brézis' conjecture was found to be false. The question has now become what conditions on a sectorial operator ensure that it has L_p-maximal regularity.

Sectorial operators owe their name to the fact that their spectrum is contained inside a sector of some angle $0 < \phi < \pi$. We can picture sectorial operators in terms of their spectrum in the complex plane as follows.



Formally, a closed operator B on X is sectorial if three properties are satisfied:

- The domain and range of B are dense
- B is one-to-one
- $\|\lambda(\lambda - B)^{-1}\| \leq C$ where $|\arg \lambda| > \phi$

We define the angle of sectoriality of B as the infimum of all angles ϕ for which the above conditions hold.

The primary example of a sectorial operator is the derivative operator on $L_p(\mathbb{R})$. It is not difficult to check using the exponential function that the spectrum of a derivative operator is the imaginary axis, and hence its sectoriality angle is $\pi/2$.

The abstract Cauchy Problem (1) can be formally written as

$$(A + B)u = h \tag{2}$$

where A is the derivative operator $(Au)(t) = \dot{u}(t)$, and B is the extension of the original operator on $L_p(X)$.

If $A(A+B)^{-1}$ is a bounded operator on $L_p(X)$ then for the solution u we have that $Au = A(A+B)^{-1}h$ is in $L_p(X)$. Thus, $\dot{u}$ is in $L_p(X)$ and B has maximal regularity. Therefore, the maximal regularity problem can be understood as a more general problem concerning the sum of two sectorial commuting operators A and B. In this abstract formulation, DaPrato and Grisvard demonstrated in 1975 [9] that maximal regularity is equivalent to the following three conditions denoted by $(*)$

- $A + B$ is closed on the intersection of the respective domains
- There is a constant $C > 0$ such that

$$\|Ax\| + \|Bx\| \leq C\|Ax + Bx\|$$

- $A + B$ is invertible if either A or B is invertible

The most widely known theorem in this context is due to Dore and Venni from 1987 [13] who proved that

> *($*$) holds if X is a UMD space and A, B have bounded imaginary powers.*

To apply this theorem to a maximal regularity problem we need only to observe that when X has UMD and $1 < p < \infty$ then the derivative operator has boundary imaginary powers.

On UMD spaces the derivative operator actually has a stronger property than bounded imaginary powers, which is H^∞-calculus. Roughly speaking, a sectorial operator A has H^∞-calculus if for any bounded analytic function f on a sector containing the spectrum of A we can define $f(A)$ as a bounded operator on X. The notion of H^∞-calculus for sectorial operators was introduced by McIntosh in 1986 [20]. Since then, H^∞-calculus has been established for many systems of differential operators (e.g. parabolic differential operators, Schrödinger operators).

In UMD spaces H^∞-calculus implies bounded imaginary powers. However, since many differential operators are known to have H^∞-calculus it seems natural to impose this stronger condition on one operator in (2) and a weaker condition on the other. This poses the question: Given that A has H^∞-calculus, what (weaker) conditions on B keep the conclusion ($*$) valid? There are examples which show that simply assuming B is sectorial does not suffice. The appropriate condition was discovered by Kalton and Weis in 2000 [18]. They replaced uniform boundedness of the family $\{\lambda(\lambda - B)^{-1}\}$ in the definition of a sectorial operator by R-boundedness. It is worth noting that the concept of R-boundedness, which is generally stronger than uniform boundedness, dates back to a paper on a different subject by Bourgain in 1983 [5]. For definition and properties of R-bounded families of operators we refer to [4], [14], [7]. A Theorem by Kalton and Weis connects R-boundedness to H^∞-calculus by saying that

> *($*$) holds if A has H^∞-calculus and the family of operators $\{\lambda(\lambda - B)^{-1}\}$ is R-bounded.*

R-boundedness is a sharp characterization in this theorem in the sense that we cannot go any weaker, and we do not need to be any stronger.

In the situation when one of the operators has H^∞-calculus the discovery of this description for the second operator was a milestone for the theory. A downside of this condition is that in a concrete situation it is typically quite difficult to verify. We would like to drop R-boundednes assumption for the operator B. But we need to pay the price in the form of making stronger assumptions on A. We were looking for a condition stronger than H^∞-calculus (which was later termed *absolute calculus*) for which the following theorem is true:

> *In case A has absolute calculus then $(*)$ holds without any extra assumptions on B.*

We were trying to find situations where R-boundedness of the operator was not required for the regularity of the solutions to the Cauchy problem. Two theorems due to Arendt, Batty and Bu [2, 1] pointed us in the right direction. They are concerning the Cauchy problem (1) with periodic boundary condition $u(0) = u(T)$. The first one is in the spirit of the result of Kalton and Weis [18]:

> Arendt and Bu (2002). *Suppose that X is UMD and B is a closed operator on X. The B has L_p-maximal regularity if and only if $\{k(ik - B)^{-1} : k \in \mathbb{Z}\}$ is* ***R-bounded****.*

This theorem holds with R-boundedness replaced by uniform boundedness only if the underlying space X is a Hilbert space.

The situation is significantly different if we consider maximal regularity with respect to Hölder spaces. We denote by $C^\alpha(X)$ the space of all periodic X-valued α-Hölder continuous functions on $[0, T]$. We say that B has C^α-maximal regularity if for each $h \in C^\alpha(X)$ the solution u of the Cauchy problem together with its derivative $\dot{u}$ are contained in $C^\alpha(X)$.

The second result characterizes regularity of the solutions in $C^\alpha(X)$ completely in terms of the resolvents of B without any restrictions on the Banach space X or an R-boundedness assumption.

> Arendt, Batty and Bu (2004). *A closed operator B has C^α-maximal regularity if and only if $\{k(ik-B)^{-1} : k \in \mathbb{Z}\}$ is* ***uniformly bounded****.*

Since R-boudnedness is not required here, the derivative operator must have stronger properties on Hölder spaces than on L_p spaces. Similar results have been obtained for Besov spaces.

There is another instance where properties of an operator improve when the underlying space changes. This happens when the operator is considered on interpolation spaces between its domain and range. A sample result in this direction is of Dore from 1999, who proved that any invertible sectorial operator on X has H^∞-calculus on interpolation spaces between its domain and X [12]. To see the connection with the theorems of Arendt, Batty and Bu, we note that for a differential operator of some type such interpolation spaces can be identified as Sobolev or Besov spaces.

In addition, Auscher, McIntosh and Nahmod (1997) showed that on Hilbert spaces we can use interpolation spaces to test for H^∞-calculus. Namely, a sectorial operator has H^∞-calculus on a Hilbert space if and only if it can be identified as an interpolation space between the domain of the operator and the range [3]. This means that if a sectorial operator A is 'nice' on X (i.e. for Hilbert spaces: has H^∞-calculus) then X has to be an interpolation space between the domain and range of A. We wish to generalize this idea to Banach spaces, but H^∞-calculus is not 'nice' enough property to make the same conclusion. Our goal is to characterize sectorial operators for the interpolation couple $(\mathrm{Ran}(A), \mathrm{Dom}(A))$ or more generally $(\mathrm{Dom}(A^{-b}), \mathrm{Dom}(A^a))$ where A^a denotes the a-th fractional power of A.

One of the methods of constructing real interpolation spaces involves K-functionals (see [6] for an exposition). The principle of K-divisibility of Brudnyi and Krugljak asserts that an intermediate space X can be realized as an interpolation space for some couple if and only if the corresponding K-functional satisfies the condition of K-monotonicity.

For a couple $(\mathrm{Dom}(A^{-b}), \mathrm{Dom}(A^a))$, we find an equivalent expression for the K-functional in terms of the norm of a specific function of the sectional operator A. Combining the principle of K-divisibility with this description of the K-functional, we arrive at the definition of *absolute calculus*. It resembles K-monotonicity but with a critical adjustment factor. The article [16], also included in this volume, explains in depth how ideas from interpolation theory apply to the study of sectorial operators. It elaborates on the history and value of this approach and how the concept of absolute calculus arises.

Absolute calculus exhibits several desired properties. First of all, it is stronger than H^∞-calculus. Secondly, if A is a sectorial operator on X and X can be realized as an interpolation space between $\mathrm{Dom}(A^{-b})$ and $\mathrm{Dom}(A^a)$ then A has absolute calculus. Note that in this way we recover the related result of Dore. Also, under a few mild assumptions we have a converse: In case A has absolute calculus on X then X can be identified as an interpolation space. Consequently, the results of McIntosh, Auscher and Nahmod mentioned above can be seen as the Hilbert space analogue. Finally, we hit one of our targets stated earlier by showing that if A has absolute calculus then $(*)$ holds without any extra assumptions on B.

In summary, our paper addresses various aspects of sectorial operators and H^∞-calculus. Some of the main motivations for developing H^∞-calculus were its natural relationships with maximal regularity of Banach space Cauchy problems and interpolation theory. We introduce a new concept called *absolute functional calculus*, which is a stronger property than H^∞-calculus, and show its relevance and power for Cauchy problems and interpolation.

The paper is organized as follows. In sections three, four, and six we give definitions, examples, preliminary results and introductory applications of absolute functional calculus to the classical theory of H^∞-calculus. We believe that these are important to understanding the concept of absolute functional calculus. Sections two and five cover abstract interpolation theory for pairs of the form $(\mathrm{Dom}(A^{-b}), \mathrm{Dom}(A^a))$, and the results obtained here provide an understanding of the relationship between functional calculus and real interpolation theory. Our main results demonstrate that absolute functional calculus is the appropriate notion for this topic, which allows us to extend a series of previous results. In sections seven and eight we focus on the study of mild and strong solutions in the abstract Cauchy problem framework. We are using the fact that certain derivative type operators on Besov spaces have an absolute functional calculus. This allows us to generalize some recent work of Arendt, Batty, and Bu on first order Cauchy problems on spaces of the form $C^\alpha(X)$.

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