LOCALIZED PRESSURE AND EQUILIBRIUM STATES

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Abstract. We introduce the notion of localized topological pressure for continuous maps on compact metric spaces. The localized pressure of a continuous potential φ is computed by considering only those (n, ε)-separated sets whose statistical sums with respect to an m-dimensional potential Φ are "close" to a given value $w \in \mathbb{R}^m$. We then establish for several classes of systems and potentials φ and Φ a local version of the variational principle. We also construct examples showing that the assumptions in the localized variational principle are fairly sharp. Next, we study localized equilibrium states and show that even in the case of subshifts of finite type and Hölder continuous potentials, there are several new phenomena that do not occur in the theory of classical equilibrium states. In particular, we construct an example with infinitely many ergodic localized equilibrium states. We also show that for systems with strong thermodynamic properties and w in the interior of the rotation set of Φ there is at least one and at most finitely many localized equilibrium states.

1. Introduction

1.1. **Motivation.** The thermodynamic formalism has been an important tool in the development of the theory of dynamical systems. Originally, this subject was primarily driven by applications in dimension theory that followed the pioneer works carried out by Ruelle, Bowen and Manning and McCluskey [6, 31, 22]. These works inspired numerous studies and generalizations with applications far beyond the sole focus on dimension. For example, pressure can be applied to obtain information about Lyapunov exponents, dimension, multifractal spectra, or natural invariant measures. We refer to [1, 26, 29, 32] for details and further references.

The main object in the thermodynamic formalism is the topological pressure, a certain functional defined on the space of observables that encodes several important quantities of the underlying dynamical system. The relation between the topological pressure and invariant measures is established by the variational principle. Namely, if $f : X \to X$ is a continuous map on a compact metric space and $\varphi : X \to \mathbb{R}$ is a continuous potential, then the topological pressure $P_{\rm top}(\varphi)$ is given by the supremum of the free energy of

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the invariant probability measures (see (12) for the precise statement). This result is powerful in part because it connects in a natural but unexpected way topological and statistical dynamics. Invariant probabilities maximizing free energy are called equilibrium states. The study of equilibrium states (existence, uniqueness and properties) has a long history and the results are widely spread in the literature, yet a complete understanding is still lacking today. We refer to [5, 7, 11, 19] for references and details.

Our focus in this paper is somewhat different. We introduce a localized version of the topological pressure where the localization results from using only those orbits in the computation of the pressure whose statistical averages with respect to a given m-dimensional potential Φ are close to a vector $w \in \mathbb{R}^m$. We then establish a version of the localized variational principle for a wide variety of systems and potentials. We also show that the assumptions in our localized variational principle are fairly sharp. Finally, we develop the theory of localized equilibrium states and study the uniqueness and finiteness of these equilibrium states. Our results significantly distinguish localized equilibrium states from the theory of classical equilibrium states.

The results in this paper are related and can be considered in some sense extensions of results in the higher dimensional multifractal analysis developed by Barreira, Saussol, Schmeling, Takens, Verbitskiy, and others (see for example [2, 3, 33]). For localizations using restrictions of the pressure to non-compact subsets we refer to [8, 27, 35] and the references therein. We will now describe our results in more detail.

1.2. **Basic definitions and statement of the results.** Let $f : X \to X$ be a continuous map on a compact metric space (X, d). We consider continuous potentials $\varphi : X \to \mathbb{R}$ and $\Phi = (\phi_1, \cdots, \phi_m) : X \to \mathbb{R}^m$. We think of φ as our target potential for computing the localized topological pressure and of Φ as the potential providing the localization. For $n \in \mathbb{N}$ and $\varepsilon > 0$, we say that $F \subset X$ is (n, ε)-separated if for all $x, y \in F$ with $x \neq y$ we have $d_n(x, y) \stackrel{\text{def}}{=} \max_{k=0,\cdots,n-1} d(f^k(x), f^k(y)) \geq \varepsilon$. Note that d_n is a metric (called Bowen metric) that induces the same topology on X as d. For $x \in X$ and $n \in \mathbb{N}$, we denote by $\frac{1}{n} S_n\Phi(x)$ the m-dimensional Birkhoff average at x of length n with respect to Φ, where

$$S_n\Phi(x) = (S_n\phi_1(x), \ldots, S_n\phi_m(x)) \tag{1}$$

and $S_n\phi_i(x) = \sum_{k=0}^{n-1} \phi_i(f^k(x))$. Given $w \in \mathbb{R}^m$ and $r > 0$ we say a set $F \subset X$ is a (n, ε, w, r)-set if F is (n, ε)-separated set and for all $x \in F$ the Birkhoff average $\frac{1}{n} S_n\Phi(x)$ is contained in the Euclidean ball $D(w, r)$ with center w and radius r. We define the *localized topological pressure* of the potential φ (with respect to Φ and w) by

$$P_{\text{top}}(\varphi, \Phi, w) = \lim_{r\to 0} \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N_\varphi(n, \varepsilon, w, r), \tag{2}$$

where

$$N_\varphi(n,\varepsilon,w,r) = \sup\left\{\sum_{x\in F} e^{S_n\varphi(x)} \,:\, F \text{ is } (n,\varepsilon,w,r)\text{-set}\right\}. \tag{3}$$

This definition is analogous to that of the classical topological pressure with the exception that we here only consider orbits with Birkhoff averages close to w. Moreover, when we omit the limit $r \to 0$ in (2) and choose r large enough that the range of Φ is contained in $D(w,r)$, then we obtain the classical topological pressure of φ.

Note that the definition of $P_{\rm top}(\varphi,\Phi,w)$ is only meaningful if $D(w,r)$ contains statistical averages with respect to Φ for infinitely many n and arbitrarily small r. We call the corresponding set of points w the *pointwise rotation set* of Φ and denote it by $\mathrm{Rot}_{Pt}(\Phi)$, that is

$$\mathrm{Rot}_{Pt}(\Phi) = \left\{ w\in\mathbb{R}^m : \forall r>0\ \forall N\ \exists n\geq N\ \exists\, x\in X:\ \frac{1}{n}S_n\Phi(x)\in D(w,r)\right\} \tag{4}$$

Next, we discuss a measure-theoretic approach to rotation sets and localized pressure. We denote by $\mathcal{M}$ the set of all Borel f-invariant probability measures on X endowed with the weak* topology. Following [15], we define the *generalized rotation set* of Φ by

$$\mathrm{Rot}(\Phi) = \{\mathrm{rv}(\mu) : \mu\in\mathcal{M}\}\,, \tag{5}$$

where $\mathrm{rv}(\mu) = \left(\int \phi_1\, d\mu,\dots,\int\phi_m\, d\mu\right)$ denotes the rotation vector of the measure μ. We call $\mathcal{M}_\Phi(w) = \{\mu\in\mathcal{M} : \mathrm{rv}(\mu) = w\}$ the rotation class of w. In [20] we study the relationship between the pointwise rotation set and generalized rotation set of Φ. In particular, we show that $\mathrm{Rot}_{Pt}(\Phi)\subset \mathrm{Rot}(\Phi)$ with strict inclusion in certain cases. We also provide criteria for the equality of the two rotation sets. We refer to the overview article [24] and to [15, 20, 37] for further details about rotation sets. For $w\in\mathrm{Rot}(\Phi)$, we define the *localized measure-theoretic pressure* of the potential φ (with respect to Φ and w) by

$$P_{\rm m}(\varphi,\Phi,w) = \sup\left\{h_\mu(f) + \int_X \varphi\, d\mu :\ \mu\in\mathcal{M}_\Phi(w)\right\}. \tag{6}$$

If we take the supremum in (6) over all invariant measures we obtain the classical measure-theoretic pressure. We now list two properties of the localized pressure which are used repeatedly throughout the text.

- The pressure function $w\mapsto P_{\rm m}(\varphi,\Phi,w)$ is concave, i.e. for any $w_1,w_2\in\mathrm{Rot}(\Phi)$ and $t\in[0,1]$ we have $P_{\rm m}(\varphi,\Phi,tw_1+(1-t)w_2)\geq tP_{\rm m}(\varphi,\Phi,w_1)+(1-t)P_{\rm m}(\varphi,\Phi,w_2)$.
- In the case when the entropy map $\mu\mapsto h_\mu(f)$ is upper semi-continuous, the pressure function $w\mapsto P_{\rm m}(\varphi,\Phi,w)$ is continuous on $\mathrm{Rot}(\Phi)$.

The concavity of the pressure function is an immediate consequence of the fact that the entropy map $\mu \mapsto h_\mu(f)$ is affine. Since the pressure function is concave it is continuous in the interior of $\mathrm{Rot}(\Phi)$. Moreover, for any boundary point w we have $\limsup_{v\to w} P_{\mathrm{m}}(\varphi,\Phi,v) \geq P_{\mathrm{m}}(\varphi,\Phi,w)$ (see e.g. [30]). To obtain the opposite inequality consider an arbitrary sequence $(v_n)_n \subset \mathrm{Rot}(\Phi)$ with $v_n \to w$ as $n \to \infty$. Let ν_n be invariant measures such that $\mathrm{rv}(\nu_n) = v_n$ and $h_{\nu_n}(f) + \int_X \varphi\, d\nu_n > P_{\mathrm{m}}(\varphi,\Phi,v_n) - \frac{1}{n}$. Let μ be the accumulation point of $(\nu_n)_n$. By passing to a subsequence if necessary we may assume that $\nu_n \to \mu$. The upper semi-continuity of the entropy map assures that $\limsup_{n\to\infty} h_{\nu_n}(f) \leq h_\mu(f)$ and thus $h_\mu(f) + \int_X \varphi\, d\mu \geq \limsup_{n\to\infty} P_{\mathrm{m}}(\varphi,\Phi,v_n)$. Since $\mathrm{rv}(\mu) = w$, we obtain

$$P_{\mathrm{m}}(\varphi,\Phi,w) \geq \limsup_{n\to\infty} P_{\mathrm{m}}(\varphi,\Phi,v_n).$$

This implies the continuity of $P_{\mathrm{m}}(\varphi,\Phi,w)$ at w.

The classical variational principle (without localization) states that the topological and the measure-theoretic versions of the pressure coincide. However, it turns out that in the case of localized pressure, the measure-theoretic and topological pressures may differ and strict inequalities can occur in both directions. This follows from the Examples 1 and 2 given in Section 3. On the other hand, the following result (see Theorem 1 in the text) gives a fairly complete description of the assumptions needed to still have a variational principle.

Theorem A. *Let $f : X \to X$ be a continuous map on a compact metric space X. Let $\varphi : X \to \mathbb{R}$ and $\Phi : X \to \mathbb{R}^m$ be continuous and let $w \in \mathrm{Rot}(\Phi)$ be such that the map $v \mapsto P_m(\varphi,\Phi,v)$ is continuous at w and $P_{\mathrm{m}}(\varphi,\Phi,w)$ is approximated by ergodic measures. Then $P_{\mathrm{top}}(\varphi,\Phi,w) = P_{\mathrm{m}}(\varphi,\Phi,w)$.*

The assumption that $P_{\mathrm{m}}(\varphi,\Phi,w)$ is approximated by ergodic measures (see Section 3 for the precise definition) cannot be dropped in Theorem A. Indeed, Example 1 does not satisfy this assumption and $P_{\mathrm{top}}(\varphi,\Phi,w) < P_{\mathrm{m}}(\varphi,\Phi,w)$ holds. On the other hand, without the assumption that $v \mapsto P_m(\varphi,\Phi,v)$ is continuous at w, Theorem A is in general not true, which is a consequence of Example 2. We recall that the continuity of $v \mapsto P_m(\varphi,\Phi,v)$ at w holds for $w \in \mathrm{int}\,\mathrm{Rot}(\Phi)$ and for all $w \in \mathrm{Rot}(\Phi)$ if the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous. In particular, this is true if f is expansive [36], a C^∞ map on a compact smooth Riemannian manifold [25] or satisfies the entropy-expansiveness (as for example certain partial hyperbolic systems [13] do). Recently, there has been significant progress in finding milder conditions that imply the upper-semicontinuity of the entropy function (see for example [9]).

We note that Theorem A holds for a wide variety of systems and potentials. In particular, Theorem A holds for systems with strong thermodynamic properties (STP) (see Section 3). A specialized version of this theorem for $C^{1+\alpha}$ conformal repellers was established in [12]. Also, in this context, a related result based on one-dimensional rotation sets can be found in [34].

Next, we present our results about localized equilibrium states. Fix $w \in \mathrm{Rot}(\Phi)$. We say $\mu \in \mathcal{M}_\Phi(w)$ is a localized equilibrium state of $\varphi \in C(X,\mathbb{R})$ (with respect to Φ and w) if

$$h_\mu(f) + \int_X \varphi \, d\mu = \sup_{\nu \in \mathcal{M}_\Phi(w)} \left(h_\nu(f) + \int_X \varphi \, d\nu \right). \tag{7}$$

This definition is analogous to that of a classical equilibrium state with the exception that we here only consider invariant measures in $\mathcal{M}_\Phi(w)$ rather than all invariant measures. Evidently, the upper semi-continuity of the entropy map guarantees the existence of at least one localized equilibrium state. Unlike in the case of classical equilibrium states, there does not need to exist an ergodic localized equilibrium state (see Example 3). In Section 4 we introduce the class of systems with strong thermodynamic properties that include subshifts of finite type, hyperbolic systems and expansive homeomorphisms with specification. These systems exhibit the strongest possible properties for classical equilibrium states. In particular, for each Hölder continuous potential φ, there exists a unique equilibrium state μ_φ (which is ergodic) and μ_φ has the Gibbs property. We show that this result does not carry over to localized equilibrium states. Indeed, in Theorem 2 we construct a Lipschitz continuous potential for a shift map that has infinitely many ergodic localized equilibrium states, none of which is Gibbs. This example is formulated for $\phi \equiv 0$ (i.e. the localized entropy) and a point w on the boundary of the $\mathrm{Rot}(\Phi)$. In Theorem B (i) (see below), we show that these phenomena do not not occur if w is in the interior of the $\mathrm{Rot}(\Phi)$.

This motivates the following definition: Let μ be a localized equilibrium state of φ (with respect to Φ and w). We say μ is an *interior localized equilibrium state* if $(\int \varphi \, d\mu, w) \in \mathrm{ri}\,\mathrm{Rot}(\varphi, \Phi)$ (where ri denotes the relative interior of the set), otherwise we say μ is a *localized equilibrium state at the boundary.* Without loss of generality we can always assume that $\dim \mathrm{Rot}(\Phi) = m$ (i.e. $\mathrm{Rot}(\Phi)$ has non-empty interior in $\mathbb{R}^m$) because otherwise we could just consider a lower dimensional affine subspace. The following result shows that interior equilibrium states still share many of the properties of classical equilibrium states.

Theorem B. *Suppose that $f : X \to X$ is a system with strong thermodynamic properties. Let φ and Φ be Hölder continuous potentials, and let $w \in \mathrm{int}\ \mathrm{Rot}(\Phi)$. Then*

- (i) *If $\dim \mathrm{Rot}(\varphi, \Phi) = m$, then there exists a unique (ergodic) localized equilibrium state at w.*
- (ii) *Suppose $\dim \mathrm{Rot}(\varphi, \Phi) = m+1$ and that all localized equilibrium states of φ are interior equilibrium states. Then the set of ergodic localized equilibrium states is non-empty and finite.*
- (iii) *Under each of the assumptions* (i) *or* (ii)*, every ergodic localized equilibrium state μ_φ is a classical equilibrium state of the potential $s\varphi + t\cdot\Phi$ for some $s \in \mathbb{R}$ and $t \in \mathbb{R}^m$.*

We note that part (i) of Theorem B holds in particular for $\varphi \equiv 0$ (and more generally if φ is cohomologous to a constant). Therefore, the assumption $w \in \mathrm{int}\,\mathrm{Rot}(\Phi)$ implies the existence of a unique localized measure of maximal entropy. Another interesting feature of Theorem B is that in both cases, (i) and (ii) the ergodic localized equilibrium state is a classical equilibrium state. This implies that if f is a subshift of finite type, a uniformly hyperbolic system or an expansive homeomorphism with specification, any ergodic localized equilibrium state is a Gibbs state.

The proof of Theorem B relies heavily on methods from the thermodynamic formalism and, in particular, on the analyticity of the topological pressure for Hölder continuous potentials. Moreover, we use results of Jenkinson [15] as key ingredients.

This paper is organized as follows: In Section 2, we review some background material. Section 3 is devoted to the proof of the localized variational principle (Theorem A) and the construction of certain examples showing that without the assumptions of Theorem A, the localized variational principle fails. Finally, in Section 4 we discuss localized equilibrium states and discover fundamental differences between the theory of classical and localized equilibrium states. In particular, we prove Theorem B for systems with strong thermodynamic properties.

2. Preliminaries

In this paper we consider deterministic discrete-time dynamical systems given by a continuous map $f : X \to X$ on a compact metric space (X, d). We are concerned with a continuous potential $\varphi : X \to \mathbb{R}$ and an m-dimensional continuous potential $\Phi = (\phi_1, \ldots, \phi_m) : X \to \mathbb{R}^m$. Consider the set $\mathcal{M}$ of all Borel f-invariant probability measures endowed with weak* topology and denote by $\mathcal{M}_E \subset \mathcal{M}$ the subset of ergodic measures. We recall the definition of the pointwise rotation set $\mathrm{Rot}_{Pt}(\Phi)$ (see (4)) and the rotation set $\mathrm{Rot}(\Phi)$ (see (5)). Similarly, the *ergodic rotation set* is defined by

$$\mathrm{Rot}_E(\Phi) = \{\mathrm{rv}(\mu) : \mu \in \mathcal{M}_E\}\,. \tag{8}$$

Rotation sets originated from Poincaré's rotation numbers for circle homeomorphisms [28]. The relation between the three different rotation sets is studied in detail in [20]. Both, $\mathrm{Rot}_{Pt}(\Phi)$ and $\mathrm{Rot}(\Phi)$ are compact and $\mathrm{Rot}(\Phi)$ is convex. We always have

$$\mathrm{Rot}_E(\Phi) \subset \mathrm{Rot}_{Pt}(\Phi) \subset \mathrm{Rot}(\Phi), \tag{9}$$

where both inclusions can be strict. The first inclusion follows from Birkhoff's Ergodic Theorem and the second is a consequence of the sequential compactness of $\mathcal{M}$ (see [20] for details).

For completeness we now recall the notion of the classical topological pressure. For $n \in \mathbb{N}$ and $\varepsilon > 0$ let

$$N_\varphi(n,\varepsilon) = \sup\left\{\sum_{x\in F} e^{S_n\varphi(x)} \,:\, F \subset X \text{ is } (n,\varepsilon)\text{-separated}\right\}. \tag{10}$$

The *topological pressure* with respect to the dynamical system (X, f) is a mapping $P_{\text{top}}(f,\cdot)\colon C(X,\mathbb{R}) \to \mathbb{R}\cup\{\infty\}$ defined by

$$P_{\text{top}}(\varphi) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log N_\varphi(n,\varepsilon). \tag{11}$$

The topological entropy of f is defined by $h_{\text{top}}(f) = P(f, 0)$. We simply write $P_{\text{top}}(\varphi)$ and h_{top} if there is no confusion about f. The topological pressure is finite if and only if the topological entropy of f is finite. We use $h_{\text{top}}(f) < \infty$ as a standing assumption in this paper. The topological pressure satisfies the well-known variational principle

$$P_{\text{top}}(\varphi) = \sup_{\mu\in\mathcal{M}}\left\{h_\mu(f) + \int_X \varphi\, d\mu\right\}. \tag{12}$$

Here $h_\mu(f)$ denotes the measure-theoretic entropy of f with respect to μ (see [36] for details). It is a straight forward conclusion that the supremum in (12) can be replaced by the supremum taken only over all $\mu \in \mathcal{M}_{\text{E}}$.

3. Localized Pressure

Our goal is to prove the local version of the variational principle, namely $P_{\text{top}}(\varphi,\Phi,w) = P_{\text{m}}(\varphi,\Phi,w)$. However, in general this equality does not hold even if the potential φ is identically zero. The following examples show that without additional assumptions we do not have even a one-sided inequality.

The first example is a dynamical system where at certain points localized topological pressure is strictly less that the localized measure-theoretic pressure. We concatenate three non-overlapping one-dimensional dynamical systems such that the entropy of the outside components is greater than the entropy of the inside one. We take the potential Φ to be the identity map and φ to be zero. Since in this case the topological pressure does not exceed the topological entropy, the concavity of the measure-theoretic pressure implies the strict inequality at the center points. What follows is the concrete construction.

Example 1. *Let $X = X_1 \cup X_2 \cup X_3$, where $X_1 = [0,1]$, $X_2 \subset [2,3]$, and $X_3 = [4,5]$. We define $f : X \to X$ to be the logistic type map on X_1 and X_3 given by*

$$f|_{X_1}(x) = 4x(1-x),\ f|_{X_3}(x) = f|_{X_1}(x-4) + 4$$

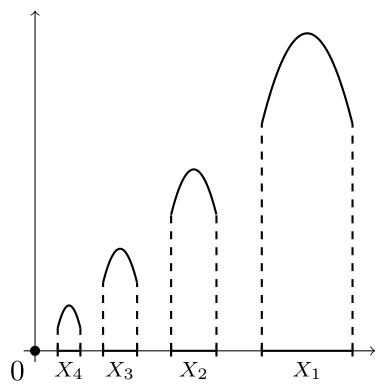
Then $h_{\text{top}}(f|_{X_1}) = h_{\text{top}}(f|_{X_3}) = \log 2$.

Whenever $f|_{X_2}$ satisfies $h_{\text{top}}(f|_{X_2}) < \log 2$ we will reach our conclusion. For example, take X_2 to be a Cantor set in the interval $[2,3]$ and f to be a homeomorphism on the Cantor set X_2 which is topologically conjugate to *a subshift whose entropy is strictly less than* $\log 2$. *One possibility is the subshift with transition matrix* $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. *We may also let* $X_2 = [2, 3]$ *and* $f|_{X_2}(x) = a(x-2)(3-x)+2$ *with* $0 < a < 4$. *In this case we also have* $h_{\rm top}(f|_{X_2}) < \log 2$.

We take the potential Φ *to be the identity map on* X. *Then for any point* $w \in \mathrm{Rot}_{Pt}(\Phi) \cap X_2$ *we have* $P_{\rm m}(0, \Phi, w) = \log 2$ *since localized measure-theoretic pressure is a concave function of* w. *However,* $P_{\rm top}(0, \Phi, w) \leq h_{\rm top}(f|_{X_2}) < \log 2$. *Therefore,* $P_{\rm top}(0, \Phi, w) < P_{\rm m}(0, \Phi, w)$.

The next example addresses the reverse inequality.

Example 2. *Consider a decreasing sequence of disjoint compact intervals* X_n *on the real line whose left end-points converge to* 0. *We define the function* f *on each* X_n *to be conjugate to the logistic map* $g(x) = 4x(1-x)$ *on* $[0, 1]$ *and maps* X_n *onto* X_n. *We also set* $f(0) = 0$. *The graph of* $f(x)$ *is shown below.*



Then $X = \cup_{n=1}^{\infty} X_n \cup \{0\}$ *is compact and* f *is continuous on* X. *Moreover, for each* n *the interval* X_n *is invariant with respect to* f. *Since* $f|_{X_n}$ *is conjugate to* $g(x) = 4x(1-x)$ *on* $[0, 1]$, *the topological entropy of* $f|_{X_n}$ *is equal to the topological entropy of* g *on* $[0, 1]$ *and therefore is* $\log 2$.

As an example of such construction consider disjoint dyadic intervals $X_n = [2^{-2n}, 2^{-2n+1}]\,(n \in \mathbb{N})$. *In this case* $f : X \to X$ *is defined in the following way.*

$$f(x) = \begin{cases} 0, & \text{if } x = 0; \\ 2^n(x - 2^{-2n})(2^{-2n+1} - x) + 2^{-2n}, & \text{if } x \in X_n. \end{cases}$$

Take the identity potential $\Phi : X \to \mathbb{R}$, $\Phi(x) = x$ *and zero potential* $\varphi(x) \equiv 0$. *For a fixed* $r > 0$ *there is* $m \in \mathbb{N}$ *such that* $X_m \subset D(0, r)$. *Then*

$$h_{\rm top}(f|_{X_m}) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \log \mathrm{card}(F_n),$$

where F_n *is a maximal* (n, ε)*-separated set of* X_m. *Since* Φ *is an identity map,* F_n *is also an* $(n, \varepsilon, 0, r)$*-set and thus* $N_\varphi(n, \varepsilon, 0, r) \geq \mathrm{card}(F_n)$. *Passing* *to the limit as* $n \to \infty$ *and* $\varepsilon \to 0$ *we see that*

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N_\varphi(n, \varepsilon, 0, r) \geq \log 2,$$

which implies that $P_{\rm top}(0, \Phi, 0) \geq \log 2$. *We actually have equality here since* $P_{\rm top}(0, \Phi, w) \leq h_{\rm top}(f) = \log 2$. *However,* $x = 0$ *is a fixed point of* f *and also an extreme point of* X. *Thus, the only invariant measure* μ *on* X *with* $\mathrm{rv}(\mu) = 0$ *is the point-mass measure at zero. Therefore,* $P_{\rm m}(0, \Phi, 0) = 0 < P_{\rm top}(0, \Phi, 0)$.

We say that $P_{\rm m}(\varphi, \Phi, w)$ is *approximated by ergodic measures* at w if there exists $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}_E$ such that $\mathrm{rv}(\mu_n) \to w$ and $h_{\mu_n}(f) + \int \varphi \, d\mu_n \to P_{\rm m}(\varphi, \Phi, w)$ as $n \to \infty$. In this case we have $w \in \mathrm{Rot}_{Pt}(\Phi)$. Indeed, for $r > 0$ there exists n such that $\mathrm{rv}(\mu_n) \in D(w, \frac{r}{2})$. The ergodicity of μ_n implies the existence of $x \in X$ such that $\frac{1}{k} S_k \Phi(x) \in D(\mathrm{rv}(\mu), \frac{r}{2})$ for arbitrary large k. Therefore, $\frac{1}{k} S_k \Phi(x) \in D(w, r)$ and thus $w \in \mathrm{Rot}_{Pt}(\Phi)$.

Theorem 1. *Let* $f : X \to X$ *be a continuous map on a compact metric space* X. *Let* $\Phi : X \to \mathbb{R}^m$ *and* $\varphi : X \to \mathbb{R}$ *be continuous and let* $w \in \mathrm{Rot}(\Phi)$ *such that the map* $v \mapsto P_m(\varphi, \Phi, v)$ *is continuous at* w *and* $P_{\rm m}(\varphi, \Phi, w)$ *is approximated by ergodic measures. Then* $P_{\rm top}(\varphi, \Phi, w) = P_{\rm m}(\varphi, \Phi, w)$.

Proof. We first show that $P_{\rm top}(\varphi, \Phi, w) \leq P_{\rm m}(\varphi, \Phi, w)$. Fix $\eta > 0$. It follows from the definition of $P_{\rm top}(\varphi, \Phi, w)$ and the continuity of $P_{\rm m}(\varphi, \Phi, v)$ at w that there exist $r > 0$ and $\varepsilon > 0$ such that

$$\left| \limsup_{n \to \infty} \frac{1}{n} \log N_\varphi(n, \varepsilon, w, r) - P_{\rm top}(\varphi, \Phi, w) \right| < \frac{\eta}{2} \tag{13}$$

and for any $v \in D(w, r) \cap \mathrm{Rot}(\Phi)$ we have

$$|P_{\rm m}(\varphi, \Phi, w) - P_{\rm m}(\varphi, \Phi, v)| < \frac{\eta}{2}. \tag{14}$$

We will now apply the method of constructing measures with large free energies which is commonly used to prove the classical variational principle. Let $\{F_n\}_{n\in\mathbb{N}}$ be (n, ε) separated sets in X such that $\frac{1}{n} S_n \Phi(x) \in D(w, r)$ for all $x \in F_n$ and $\sum_{x \in F_n} e^{S_n \varphi(x)} > \frac{1}{2} N_\varphi(n, \varepsilon, w, r)$. Let ν_n be the atomic measure concentrated on F_n given by the formula

$$\nu_n = \left(\sum_{x \in F_n} e^{S_n \varphi(x)} \right)^{-1} \sum_{x \in F_n} e^{S_n \varphi(x)} \delta_x, \tag{15}$$

where δ_x denotes the Dirac measure supported on x. Consider a sequence of measures $\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} \nu_n \circ f^{-k}$ and let μ be a weak* accumulation point of (μ_n). Then (see [36] or [18, Section 4.5]) μ is f-invariant and satisfies

$$\limsup_{n \to \infty} \frac{1}{n} \log \sum_{x \in F_n} e^{S_n \varphi(x)} \leq h_\mu(f) + \int_X \varphi \, d\mu. \tag{16}$$

We conclude that

$$\begin{aligned} P_{\rm top}(\varphi,\Phi,w) &\le \limsup_{n\to\infty}\frac{1}{n}\log N_\varphi(n,\varepsilon,w,r)+\frac{\eta}{2}\\ &=\limsup_{n\to\infty}\frac{1}{n}\log\sum_{x\in F_n}e^{S_n\varphi(x)}+\frac{\eta}{2}\\ &\le P_{\rm m}(\varphi,\Phi,{\rm rv}(\mu))+\frac{\eta}{2}. \end{aligned} \tag{17}$$

Note that $\mathrm{rv}(\mu)\in D(r,w)$ by the construction of μ. It follows from the last inequality and (14) that $P_{\rm top}(\varphi,\Phi,w)\le P_{\rm m}(\varphi,\Phi,w)+\eta$. Since η was arbitrary, we obtain the desired inequality $P_{\rm top}(\varphi,\Phi,w)\le P_{\rm m}(\varphi,\Phi,w)$.

Now we turn our attention to the opposite inequality. Let $\eta>0$ be arbitrary. As before, we fix $r_0>0$ and $\varepsilon_0>0$ such that for any $0<\varepsilon<\varepsilon_0$

$$\left|\limsup_{n\to\infty}\frac{1}{n}\log N_\varphi(n,\varepsilon,w,r_0)-P_{\rm top}(\varphi,\Phi,w)\right|<\frac{\eta}{2}. \tag{18}$$

Since Φ is uniformly continuous on X we may assume that ε_0 is chosen small enough so that for any $n\in\mathbb{N}$ and $x_1,x_2\in X$ with $d_n(x_1,x_2)<\varepsilon_0$ we have

$$\left|\frac{1}{n}S_n\Phi(x_1)-\frac{1}{n}S_n\Phi(x_2)\right|\le\frac{r_0}{3}. \tag{19}$$

Since $P_{\rm m}(\varphi,\Phi,w)$ is approximated by ergodic measures, there exists $\mu\in\mathcal{M}_E$ such that

$$|\mathrm{rv}(\mu)-w|<\frac{r_0}{3}\quad\text{and}\quad P_{\rm m}(\varphi,\Phi,w)-\frac{\eta}{4}<h_\mu(f)+\int\varphi\,d\mu. \tag{20}$$

There is a generalization of Katok's characterization of the measure-theoretic entropy in terms of ergodic measures to the concept of topological pressure derived in [14]. See [17] for the original approach. We are using the following set up: Fix $0<\delta<1$. We say that E is an (n,ε)-spanning set for $Y\subset X$ if $Y\subset\cup_{x\in E}B_n(x,\varepsilon)$. Denote by $Q_\varphi(n,\varepsilon,\mu,\delta)=\inf\left\{\sum_{x\in E}e^{S_n\varphi(x)}\right\}$, where the infimum is taken over all (n,ε)-spanning sets E of a set of μ-measure more than or equal to $1-\delta$. Then

$$h_\mu(f)+\int\varphi\,d\mu=\lim_{\varepsilon\to0}\liminf_{n\to\infty}\frac{1}{n}\log Q_\varphi(n,\varepsilon,\mu,\delta). \tag{21}$$

There exists a decreasing sequence of strictly positive numbers $\varepsilon_i<\varepsilon_0$, ($i\in\mathbb{N}$) with $\lim_{i\to0}\varepsilon_i=0$ and corresponding sequences of (n,ε_i)-spanning sets $E_n(\varepsilon_i)$ $(n\in\mathbb{N})$ such that

$$\sum_{x\in E_n(\varepsilon_i)}e^{S_n\varphi(x)}<2Q_\varphi(n,\varepsilon_i,\mu,\delta). \tag{22}$$

We may assume that each $E_n(\varepsilon_i)$ is a minimal spanning set with respect to the inclusion. Since μ is ergodic, the basin of μ defined by

$$\mathcal{B}(\mu)=\left\{x\in X:\frac{1}{n}\sum_{k=1}^{n-1}\delta_{f^k(x)}\to\mu\text{ as }n\to\infty\right\} \tag{23}$$

is a set of full μ-measure by Birkhoff's Ergodic Theorem. We define

$$\mathcal{B}_{n,\frac{r_0}{3}}(\mu) = \left\{ x \in \mathcal{B}(\mu) : \left| \frac{1}{l} S_l \Phi(x) - \mathrm{rv}(\mu) \right| < \frac{r_0}{3} \text{ for all } l \geq n \right\}. \tag{24}$$

Since $(\mathcal{B}_{n,\frac{r_0}{3}}(\mu))_{n\in\mathbb{N}}$ is an increasing sequence of Borel sets whose union is a set of full μ-measure, we conclude that $\lim_{n\to\infty} \mu(\mathcal{B}_{n,\frac{r_0}{3}}(\mu)) = 1$. Consider the sequence of sets

$$\tilde{E}_n(\varepsilon_i) = \left\{ x \in E_n(\varepsilon_i) : B_n(x, \varepsilon_i) \cap \mathcal{B}_{n,\frac{r_0}{3}}(\mu) \neq \varnothing \right\}. \tag{25}$$

It follows from (19) and (20) that for any $x \in \tilde{E}_n(\varepsilon_i)$ we have $\frac{1}{n} S_n \Phi(x) \in D(w, r_0)$. When n is sufficiently large, $\tilde{E}_n(\varepsilon_i)$ is a spanning set for a set of μ-measure greater than $1 - \delta'$ where $\delta < \delta' < 1$. Therefore,

$$Q_\varphi(n, \varepsilon_i, \mu, \delta') \leq \sum_{x\in\tilde{E}_n(\varepsilon_i)} e^{S_n\varphi(x)} \leq \sum_{x\in E_n(\varepsilon_i)} e^{S_n\varphi(x)} < 2Q_\varphi(n, \varepsilon_i, \mu, \delta). \tag{26}$$

It follows from the $5r$-covering theorem (see for example [23]) that we can select a set of points $F_n(\varepsilon_i)$ from the set of $\tilde{E}_n(\varepsilon_i)$ such that the set $\{B_n(x, \varepsilon_i) : x \in F_n(\varepsilon_i)\}$ is pairwise disjoin and $F_n(\varepsilon_i)$ is $(5\varepsilon_i, n)$ spanning for the above set of μ-measure greater than $1 - \delta'$. Then we can rewrite the previous set of inequalities

$$Q_\varphi(n, 5\varepsilon_i, \mu, \delta') \leq \sum_{x\in F_n(\varepsilon_i)} e^{S_n\varphi(x)} \leq \sum_{x\in\tilde{E}_n(\varepsilon_i)} e^{S_n\varphi(x)} < 2Q_\varphi(n, \varepsilon_i, \mu, \delta). \tag{27}$$

Since

$$\begin{aligned} h_\mu(f) + \int \varphi\, d\mu &= \lim_{i\to\infty} \liminf_{n\to\infty} \frac{1}{n} \log Q_\varphi(n, \varepsilon_i, \mu, \delta) \\ &= \lim_{i\to\infty} \liminf_{n\to\infty} \frac{1}{n} \log Q_\varphi(n, 5\varepsilon_i, \mu, \delta') \end{aligned}$$

for i large enough we obtain

$$\left(h_\mu(f) + \int \varphi\, d\mu \right) - \frac{\eta}{4} < \liminf_{n\to\infty} \frac{1}{n} \log \sum_{x\in F_n(\varepsilon_i)} e^{S_n\varphi(x)}. \tag{28}$$

Combining inequality (28) with (18), (20) and the fact that $\sum_{x\in F_n(\varepsilon_i)} e^{S_n\varphi(x)} \leq N_\varphi(n, \varepsilon_i, w, r_0)$ we obtain

$$P_{\mathrm{m}}(\varphi, \Phi, w) - \frac{\eta}{2} < P_{\mathrm{top}}(\varphi, \Phi, w) + \frac{\eta}{2}. \tag{29}$$

Since η was arbitrary, this concludes the proof of the theorem. □

Corollary 1. *Let $f : X \to X$ be a continuous map on a compact metric space X, such that the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous and the set of ergodic measures is dense in $\mathcal{M}$. Then for any continuous* *potentials* $\Phi : X \to \mathbb{R}^m$ *and* $\varphi : X \to \mathbb{R}$ *and all* $w \in \mathrm{Rot}(\Phi)$ *we have* $P_{\mathrm{top}}(\varphi, \Phi, w) = P_{\mathrm{m}}(\varphi, \Phi, w)$.

In particular, if (X, f) is a topological mixing subshift of finite type, an expensive system with specification or an Axiom A basic set the variational principle holds for all continuous potentials and all points in the rotation set.

Remarks. (i) *Note that whenever* $w \in \mathrm{Rot}_{Pt}(\Phi) \cap X_2$ *in Example 1 then* $P_{\mathrm{m}}(0, \Phi, w)$ *cannot be approximated by ergodic measures. Indeed, for any* $\mu \in \mathcal{M}_E$ *with* $w = \mathrm{rv}(\mu) \in X_2$ *we have* $\mu(X_1) = \mu(X_3) = 0$*; thus,* $h_\mu(f) \leq h_{\mathrm{top}}(f|_{X_2}) < \log 2$ *follows from the variational principle.*
(ii) *In Example 2 we observe that the function* $w \mapsto P_m(0, \Phi, w)$ *is not continuous at* $w = 0$*. Let* μ_n *be the entropy maximizing ergodic measures on* X_n*. We have* $\mathrm{rv}(\mu_n) \to 0,\ P_m(0, \Phi, \mathrm{rv}(\mu_n)) = \log 2$ *and* $P_m(0, \Phi, 0) = 0$.

4. Equilibrium states

Let $f : X \to X$ be a continuous map on a compact metric space and let $\Phi = (\phi_1, \ldots, \phi_m) \in C(X, \mathbb{R}^m)$. Fix $w \in \mathrm{Rot}(\Phi)$. We recall the definition of $\mu \in \mathcal{M}_\Phi(w)$ being a localized equilibrium state of $\varphi \in C(X, \mathbb{R})$ with respect to Φ and w in (7).

We say that the entropy map is upper semi-continuous at $w \in \mathrm{Rot}(\Phi)$ if for every $(\mu_n)_n \subset \mathcal{M}$ with $\mathrm{rv}(\mu_n) \to w$ and every accumulation point μ of $(\mu_n)_n$ we have $\limsup_{n\to\infty} h_{\mu_n}(f) \leq h_\mu(f)$. Note that if the entropy map is upper semi-continuous at w then there exists for each $\varphi \in C(X, \mathbb{R})$ at least one localized equilibrium state of φ. The following example shows that the existence of a localized equilibrium state does in general not imply the existence of an ergodic localized equilibrium state. This differs from the theory of classical equilibrium states where the existence of an equilibrium state always guarantees the existence of an ergodic equilibrium state (see [36]).

Example 3. *Let* $a, b, c, d \in \mathbb{R}$ *with* $a < b < c < d$*. Let* $X = [a, b] \cup [c, d]$ *and* $f : X \to X$ *be a continuous transformation with an upper semi-continuous entropy map* $\mu \mapsto h_\mu(f)$ *such that* $f([a, b]) \subset [a, b]$ *and* $f([c, d]) \subset [c, d]$*. Moreover, we assume that* $f(a) = a$*,* $f(d) = d$*, and* $h_{\mathrm{top}}(f|_{[a,b]}) = h_{\mathrm{top}}(f|_{[c,d]}) \neq 0$*. Consider the potentials* $\Phi = \mathrm{id}_X$ *and* $\varphi \equiv 0$*. Since* $\delta_a, \delta_d \in \mathcal{M}$ *the convexity of* $\mathrm{Rot}(\Phi)$ *implies* $\mathrm{Rot}(\Phi) = [a, d]$*. Any* $w \in (b, c)$ *can be written as* $w = \alpha \mathrm{rv}(\mu_1) + (1 - \alpha)\mathrm{rv}(\mu_2)$*, where* $\alpha \in (0, 1)$ *and* μ_1, μ_2 *are ergodic entropy maximizing measures on* $[a, b]$ *and* $[c, d]$ *respectively. It follows that the measure* $\mu = \alpha\mu_1 + (1 - \alpha)\mu_2$ *is a localized equilibrium state of* φ *with respect to* Φ *and* w*. However, the set* $\mathcal{M}_\Phi(w)$ *does not contain any ergodic measure.*

We will see that even in the case of systems satisfying the strongest possible existence and uniqueness results for classical equilibrium states the situation for localized equilibrium states is rather different. We now introduce the class of systems with strong thermodynamic properties.

4.1. **Systems with strong thermodynamic properties.** We say $f : X \to X$ has strong thermodynamic properties (which we abbreviate by (STP)) if the following conditions hold:

1. $h_{\text{top}}(f) < \infty$;
2. The entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous;
3. The map $\varphi \mapsto P_{\text{top}}(f, \varphi)$ is real-analytic on $C^\alpha(X, \mathbb{R})$;
4. Each potential $\varphi \in C^\alpha(X, \mathbb{R})$ has a unique equilibrium measure μ_φ such that $P(\varphi) = h_{\mu_\varphi}(f) + \int \varphi\, d\mu_\varphi$. Furthermore, μ_φ is ergodic and given $\psi \in C^\alpha(X, \mathbb{R})$ we have
$$\frac{d}{dt} P_{\text{top}}(f, \varphi + t\psi)\Big|_{t=0} = \int_X \psi\, d\mu_\varphi. \tag{30}$$
5. For each φ, $\psi \in C^\alpha(X, \mathbb{R})$ we have $\mu_\varphi = \mu_\psi$ if and only if $\varphi - \psi$ is cohomologous to a constant.
6. For each φ, $\psi \in C^\alpha(X, \mathbb{R})$ and $t \in \mathbb{R}$ we have
$$\frac{d^2}{dt^2} P_{\text{top}}(f, \varphi + t\psi) \geq 0, \tag{31}$$
with equality if and only if ψ is cohomologous to a constant.

Note that for several classes of systems properties (3)-(6) hold even for a wider class of potentials, namely for potentials with summable variation (see for example [15]). For simplicity, we restrict our considerations to Hölder continuous potentials.

Some examples of systems with strong thermodynamic properties are expansive homeomorphisms with specification which include topological mixing two-sided subshifts of finite type as well as diffeomorphisms with a locally maximal topological mixing hyperbolic set, see [7, 11, 18, 32]. We note that in all these examples the measure μ_φ in property (4) is a Gibbs measure.

4.2. **Interior and boundary equilibrium states.** From now on we assume that f has strong thermodynamic properties, $\Phi : X \to \mathbb{R}^m$ and $\dim \mathrm{Rot}(\Phi) = m$. Recall that if Φ is Hölder continuous then $\dim \mathrm{Rot}(\Phi) = m$ is equivalent to the condition that no non-trivial linear combination $t \cdot \Phi = t_1\phi_1 + \ldots + t_m\phi_m$ is cohomologous to a constant. For $A \subset \mathbb{R}^l$ we define the relative interior of A (denoted by ri A) as the interior of A considered as a subset of the smallest affine subspace of $\mathbb{R}^l$ containing A. In particular, if A has non-empty interior then the relative interior and the interior of A coincide.

Definition 1. *Suppose $\mu \in \mathcal{M}_\Phi(w)$ is a localized equilibrium state of $\varphi \in C(X, \mathbb{R})$ with respect to Φ and w. We say that μ is an interior equilibrium state if $(\int \varphi d\mu, w) \in \mathrm{ri}\ \mathrm{Rot}(\varphi, \Phi)$. Otherwise, call μ a localized equilibrium state at the boundary.*

We note that $\dim \mathrm{Rot}(\varphi, \Phi) = m$ if and only if either φ is cohomolous to a constant or φ is cohomologous to some nontrivial linear combination of Φ. In this situation we say that a localized equilibrium state of φ with respect to Φ and w is a localized measure of maximal entropy at w.

Next, we introduce some concepts about shift maps that will be used later on. Let $d \in \mathbb{N}$ and let $\mathcal{A} = \{0, \cdots, d-1\}$ be a finite alphabet in d symbols. The (one-sided) shift space X on the alphabet $\mathcal{A}$ is the set of all sequences $x = (x_n)_{n=1}^{\infty}$ where $x_n \in \mathcal{A}$ for all $n \in \mathbb{N}$. We endow X with the Tychonov product topology which makes X a compact metrizable space. For example, given $0 < \alpha < 1$ it is easy to see that

$$d(x, y) = d_\alpha(x, y) = \alpha^{\inf\{n \in \mathbb{N}:\ x_n \neq y_n\}} \tag{32}$$

defines a metric which induces the Tychonov product topology on X. The shift map $f : X \to X$ (defined by $f(x)_n = x_{n+1}$) is a continuous d to 1 map on X. If $Y \subset X$ is a non-empty closed f-invariant set we say that $f|_Y$ is a sub-shift. In particular, for a $d \times d$ matrix A with values in $\{0, 1\}$ we define $X_A = \{x \in X : A_{x_n, x_{n+1}} = 1\}$. It is easy to see that X_A is a closed (and therefore compact) f-invariant set and we say that $f|_{X_A}$ a subshift of finite type. A subshift of finite type is (topologically) mixing if A is aperiodic, that is, if there exists $n \in \mathbb{N}$ such that $A^n_{i,j} > 0$ for all $i, j \in \mathcal{A}$.

Let (Y, f) be a one-sided subshift. Given $x \in Y$ we write $\pi_n(x) = (x_1, \cdots, x_n)$. Moreover, for $\tau = (\tau_1, \cdots, \tau_n) \in \mathcal{A}^n$ we denote by $\mathcal{C}(\tau) = \{x \in Y : x_1 = \tau_1, \cdots, x_n = \tau_n\}$ the cylinder generated by τ and the element $\mathcal{O}(\tau) = (\tau_1, ..., \tau_n, \tau_1, ..., \tau_n, ...)$ the periodic orbit generated by τ. In this case n is referred to as the length of the cylinder or the orbit respectively.

We now show that in the case of shift maps there may exist infinitely many ergodic localized equilibrium states.

Theorem 2. *Let (X, f) be the full one-sided shift on $\{0, 1, 2, 3\}^{\mathbb{N}}$. Then there exists a Lipschitz potential $\Phi : X \to \mathbb{R}$ and a point $w \in \partial \mathrm{Rot}(\Phi)$ such that there are infinitely many ergodic localized measures of maximum entropy at w.*

Proof. We split the alphabet into two sets $S_1 = \{0, 1\}$ and $S_2 = \{2, 3\}$. Next we define certain subsets of X:

- X_0 is the set of all $x = (x_n) \in X$ such that there is at most one index n with $x_n \in S_2$.

- For $k \in \mathbb{N}, k \geq 3$ we define X_k as the set of all $x = (x_n) \in X$ such that there exists $i = i(x) \leq k$ with

  $x_n \in S_2$ whenever $n \equiv i \mod k$ and $x_n \in S_1$ otherwise.

It follows from the definitions that X_0 and X_k are f-invariant sets. We define

$$X_\infty = \overline{\bigcup_{k \geq 3} X_k}. \tag{33}$$

We claim that $X_\infty = \bigcup_{k\geq 3} X_k \cup X_0$. To prove the claim we first observe that $X_0 \subset X_\infty$. To see this let $x \in X_0$ and assume there is j with $x_j \in S_2$. Then for all $k \geq \max\{j, 3\}$ exists $x(k) \in X_k$ with $\pi_k(x(k)) = \pi_k(x)$. If there is no such j then there exists $x(k) \in X_{k+1}$ with $\pi_k(x(k)) = \pi_k(x)$. In either case it follows that $x(k) \to x$ as $k \to \infty$.

To prove the claim it suffices to show that $\bigcup_{k\geq 3} X_k \cup X_0$ is closed or, equivalently, that its complement $\bigcap_{k\geq 3} X_k^c \cap X_0^c$ is open. Pick any $x \in \bigcap_{k\geq 3} X_k^c \cap X_0^c$. Since $x \in X_0^c$ there are at least two coordinates of x which are in S_2. Denote by x_i and x_j the first and second such coordinate respectively. Then the cylinder $\mathcal{C}(\pi_j(x)) \subset X_k^c$ for $k \neq j - i$. In the case $j - i < 3$, we have

$$\mathcal{C}(\pi_j(x)) \subset X_k^c \subset \bigcap_{k\geq 3} X_k^c \cap X_0^c.$$

Now suppose that $j - i \geq 3$. Since $x \in X_{j-i}^c$ there must be an index $l > j$ for which we have one of the following

- $x_l \in S_2$ with $l \mod (j - i) \neq i$
- $x_l \in S_1$ with $l \mod (j - i) = i$.

In either case $\mathcal{C}(\pi_l(x)) \subset X_{j-i}^c$. Therefore, for any $x \in \bigcap_{k\geq 3} X_k^c \cap X_0^c$ there is an open neighbourhood $\mathcal{C}(\pi_l(x)) \subset \bigcap_{k\geq 3} X_k^c \cap X_0^c$. This completes the proof of the claim. A simplified version of this argument shows that X_0 and $X_k,\ k \geq 3$ are also closed subsets of X.

Next, we compute the topological entropy of $(X_\infty, f|_{X_\infty})$. Since X_∞ is a subshift, we need to estimate the logarithmic growth of the number of different segments of length n appearing in the sequences $x \in X_\infty$. Suppose $\tau = (\tau_1, \tau_2, ..., \tau_n)$ is such a segment. There are $(n + 1)2^n$ possibilities if τ has at most one coordinate $\tau_i \in S_2$. If the segment τ has more than one coordinate in S_2, it must come from an element in one of the $X_k,\ k \geq 3$. Moreover, k is uniquely determined by the distance between any two consecutive coordinates in τ which are from S_2. There are at most $\frac{1}{2}n(n-1)$ choices for such coordinates. Therefore, there are at most $\frac{1}{2}n(n-1)2^n$ segments of length n with two or more coordinates in S_2. We obtain

$$\begin{aligned} h_{\mathrm{top}}(f|_{X_\infty}) &= \lim_{n\to\infty} \frac{\log \operatorname{card}\{\pi_n(x) : x \in X_\infty\}}{n} \\ &= \lim \frac{\log(2^n + n2^n + \frac{1}{2}n(n-1)2^n)}{n} \\ &= \log 2. \end{aligned} \tag{34}$$

Analogously, one can show that $h_{\mathrm{top}}(f|_{X_k}) = \log 2$ for all $k \geq 3$. Since $f|_{X_k}$ is expansive there exists an ergodic measure of maximal entropy μ_k on X_k, in particular $h_{\mu_k}(f|_{X_\infty}) = \log 2$.

We define the potential $\Phi : X \mapsto \mathbb{R}$ as $\Phi(x) = d_\alpha(x, X_\infty)$. It is easy to verify that Φ is Lipschitz continuous and $\mathrm{Rot}(\Phi) = [0, \alpha^2]$.

Consider $\nu \in \mathcal{M}$ with $\mathrm{rv}(\nu) = 0$. We claim that $\nu(X_\infty) = 1$. To prove the claim we define

$$B_n = \left\{ x \in X : d_\alpha(x, X_\infty) \geq \frac{1}{n} \right\} \tag{35}$$

for all $n \in \mathbb{N}$. Note that $(B_n)_n$ is an increasing sequence of sets with $\bigcup_n B_n = X_\infty^c$. Hence $\nu(X_\infty^c) = \lim_{n\to\infty} \nu(B_n)$. If $\nu(X_\infty^c) \neq 0$ then there exists $n \in \mathbb{N}$ with $\nu(B_n) > 0$. Thus

$$\mathrm{rv}(\nu) = \int_X \Phi \, d\nu \geq \int_{B_n} \Phi \, d\nu \geq \frac{1}{n} \nu(B_n) > 0, \tag{36}$$

which is a contradiction. Since $f|_{X_\infty}^{-1}(A) = f^{-1}(A) \cap X_\infty$ for any Borel set A and $\nu(X_\infty) = 1$, it follows that ν is $f|_{X_\infty}$ invariant. Therefore, $h_\nu(f|_{X_\infty}) \leq h_{\mathrm{top}}(f|_{X_\infty}) = \log 2$. We obtain

$$\sup\{h_\nu(f) : \nu \in \mathcal{M}_\Phi(0)\} = \log 2. \tag{37}$$

Since $\mathrm{rv}(\mu_k) = 0$ and $h_{\mu_k}(f) = \log 2$ for all $k \geq 3$ each of the measures μ_k is an ergodic localized measure of maximal entropy at $0 \in \partial\mathrm{Rot}(\Phi)$.

□

Next, we consider interior localized equilibrium states. Recall our standing assumptions that $\Phi = (\phi_1, \cdots, \phi_m) \in C^\delta(X, \mathbb{R}^m)$ for some $\delta > 0$ and that $\dim \mathrm{Rot}(\Phi) = m$ (i.e. no non-trivial linear combination $t \cdot \Phi = (t_1, \cdots, t_m) \cdot \Phi$ is cohomologous to a constant). Let $\varphi \in C^\delta(X, \mathbb{R})$. We first consider the case $\dim \mathrm{Rot}(\varphi, \Phi) = m$. As noted before, this means that either φ is cohomologous to a constant or φ is cohomologous to some non-trivial linear combination $t \cdot \Phi$. It follows from the convexity of $\mathrm{Rot}(\varphi, \Phi)$ that $I_w \overset{\mathrm{def}}{=} \{\int \varphi \, d\mu : \mu \in \mathcal{M}_\Phi(w)\}$ is a singleton. In particular, $\mu \in \mathcal{M}_\Phi(w)$ is a localized equilibrium state of φ with respect to Φ and w if and only if μ is a localized measure of maximal entropy at w. For $t = (t_1, \cdots, t_m)$ let us denote by μ_t the (classical) equilibrium state of the potential $t \cdot \Phi$ (which is well-defined by property 4 of (STP)). In [20] we proved the following result.

Theorem 3. *Let $w \in \mathrm{int}\ \mathrm{Rot}(\Phi)$ and assume I_w is a singleton. Then there exits a unique localized measure of maximal entropy μ at w. Moreover, $\mu = \mu_t$ for some uniquely defined $t \in \mathbb{R}^m$.*

We now consider the case $\dim \mathrm{Rot}(\varphi, \Phi) = m + 1$. Let $w \in \mathrm{int}\ \mathrm{Rot}(\Phi)$. It follows from the compactness and the convexity of $\mathrm{Rot}(\varphi, \Phi)$ that there exist $a_w < b_w$ such that

$$I_w = \left\{ \int \varphi \, d\mu : \mu \in \mathcal{M}_\Phi(w) \right\} = \mathrm{Rot}(\varphi, \Phi) \cap \mathbb{R} \times \{w\} = [a_w, b_w]. \tag{38}$$

For $(s, t) = (s, t_1, \cdots, t_m) \in \mathbb{R}^{m+1}$ let $\mu_{s,t}$ denote the uniquely defined (classical) equilibrium measure of the potential $s\varphi + t \cdot \Phi$. It follows that

$$(s, t) \mapsto h_{\mu_{s,t}}(f) + \int s\varphi + t \cdot \Phi \, d\mu_{s,t} = P_{\mathrm{top}}(f, s\varphi + t \cdot \Phi) \tag{39}$$

is real-analytic (see proporty 3 of (STP)).

In [20] we showed that the map $F:\mathbb{R}\times\mathbb{R}^m\to \text{int Rot}(\varphi,\Phi)$ defined by

$$F(s,t)=\left(\int\varphi\ d\mu_{s,t},\int\phi_1\ d\mu_{s,t},\cdots,\int\phi_m\ d\mu_{s,t}\right) \tag{40}$$

is a real-analytic diffeomorphism and that $\mu_{s,t}$ is the unique measure satisfying

$$h(s,t)\stackrel{\text{def}}{=}h_{\mu_{s,t}}(f)=\sup\{h_\nu(f):\text{rv}(\nu)=F(s,t)\}. \tag{41}$$

Moreover, the map $(s,t)\mapsto h(s,t)$ is real-analytic. For $\alpha\in(a_w,b_w)$ we write $g(\alpha)=g_w(\alpha)=F^{-1}(\cdot,w)(\alpha)$.

Proposition 1. *Let $w\in$ int Rot(Φ). Then the map $\alpha\mapsto g(\alpha)$ is a real-analytic diffeomorphism onto its image and $\mu_{g(\alpha)}$ is the unique measure satisfying*

$$h_{\mu_{g(\alpha)}}+\int\varphi\ d\mu_{g(\alpha)}=\sup\left\{h_\mu(f)+\int\varphi\ d\mu:\mu\in\mathcal{M}_\Phi(w),\int\varphi\ d\mu=\alpha\right\}. \tag{42}$$

In particular, if μ is an interior ergodic localized equilibrium state of φ with respect to Φ and w, then there exists a unique $\alpha\in(a_w,b_w)$ with $\mu=\mu_{g(\alpha)}$.

Proof. The statement is a direct consequence of (38),(40) and (41). □

Finally, we present our main result about interior localized equilibrium states.

Theorem 4. *Supose that all localized equilibrium states of φ with respect to Φ and w are interior equilibrium states. Then there exists at least one and at most finitely many ergodic localized equilibrium states of φ with respect to Φ and w. All these ergodic localized equilibrium states are classical equilibrium states.*

Proof. Since there exists a localized equilibrium state of φ, we may conclude from Proposition 1 the existence of an ergodic localized equilibrium state of φ with respect to Φ and w. Suppose there exist infinitely many ergodic localized equilibrium states of φ. Again by Proposition 1 there exists a pairwise disjoint sequence $(\alpha_k)_{k\in\mathbb{N}}\subset(a_w,b_w)$ such that each $\mu_{g(\alpha_k)}$ is an ergodic localized equilibrium state of φ. Let μ be a weak* accumulation point of the measures $\mu_{g(\alpha_k)}$. It follows that μ is also a localized equilibrium state of φ. Recall the assumption that there are no localized equilibrium states at the boundary. Thus, Proposition 1 implies that $\mu=\mu_{g(\alpha')}$ for some $\alpha'\in(a_w,b_w)$. We conclude that the real-analytic function $\alpha\mapsto h_{\mu_{g(\alpha)}}+\int\varphi\ d\mu_{g(\alpha)}$ is constant on a non-discrete subset of (a_w,b_w). Hence, $\alpha\mapsto h_{\mu_{g(\alpha)}}+\int\varphi\ d\mu_{g(\alpha)}$ is constant by the identity theorem. Therefore, $\mu_{g(\alpha)}$ is a localized equilibrium state of φ with respect to Φ and w for every $\alpha\in(a_w,b_w)$. But this implies that there must exist a localized equilibrium state of φ with respect to Φ and w at the boundary which is a contradiction.

□

References

[1] L. Barreira and K. Gelfert, *Dimension estimates in smooth dynamics: a survey of recent results*, Ergodic Theory and Dynamical Systems **31** (2011), 641-671.

[2] L. Barreira, B. Saussol and J. Schmeling *Higher-dimensional multifractal analysis*, J. Math. Pures Appl. **9** (2002), 67-91.

[3] L. Barreira and B. Saussol, *Variational principles and mixed multifractal spectra*, Trans. Amer. Math. Soc. **353** (2001), 3919-3944.

[4] T. Bousch, *Le poisson n'a pas d'aretes*, Annales de l'Institut Henri Poincaré (probabilités et statistiques) **36** (2000), 489–508.

[5] R. Bowen, Equilibrium states and the ergodic theory of Anosov diffeomorphisms, Lecture Notes in Math. **470**, Springer-Verlag, Berlin, 1975.

[6] R. Bowen, *Hausdorff dimension of quasicircles*, Inst. Hautes Études Sci. Publ. Math. **50** (1979), 11–25.

[7] R. Bowen, *Some systems with unique equilibrium states*, Math. Systems Theory **8** (1974/75), 193-202.

[8] V. Climenhaga *Topological pressure of simultaneous level sets*, Nonlinearity **26**, (2013), 241-268.

[9] V. Climenhaga and Thompson, *Equilibrium states beyond specification and the Bowen property*, Journal London Mathematical Society (2) **87** (2013), 401-427.

[10] M. Denker, C. Grillenberger and- K. Sigmund, *Ergodic theory on compact spaces*, Lecture Notes in Mathematics, Vol. 527. Springer-Verlag, Berlin-New York, 1976. iv+360 pp.

[11] N. Haydn and D. Ruelle, *Equivalence of Gibbs and equilibrium states for homeomorphisms satisfying expansiveness and specification*, Comm. Math. Phys. **148** (1992), 155-167.

[12] D. J. Feng, K. S. Lau and J. Wu, *Ergodic limits on the conformal repeller*, Adv. Math. **169** (2002), 58–91.

[13] T. Fisher, L. Diaz, M. Pacifico and J. Vieitez, *Entropy-expansiveness for partially hyperbolic diffeomorphisms*, Discrete Contin. Dyn. Syst. **32** (2012), 4195-4207.

[14] L. He, J. Lv and L. Zhou, *Definition of measure-theoretic pressure using spanning sets*, Acta Math. Sinica, English series **20** (2004), 709–718.

[15] O. Jenkinson, *Rotation, entropy, and equilibrium states*, Trans. Amer. Math. Soc. **353** (2001), 3713–3739.

[16] O. Jenkinson, *Geometric Barycentres of Invariant Measures for Circle Maps*, Ergodic Theory and Dynamical Systems **21** (2001), 511–532.

[17] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Publ. Math., Inst. Hautes Étud. Sci. **51** (1980), 137–173.

[18] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems. With a supplementary chapter by Katok and Leonardo Mendoza*, Encyclopedia of Mathematics and its Applications, **54**, Cambridge University Press, Cambridge, 1995.

[19] G. Keller, *Equilibrium states in ergodic theory*, London Mathematical Society Student Texts, **42**, Cambridge University Press, Cambridge, 1998. x+178 pp.

[20] T. Kucherenko and C. Wolf, *Geometry and entropy of generalized rotation sets*, Israel Journal of Mathematics, to appear.

[21] T. Kucherenko and C. Wolf, *On finitelness of localized equilibrium states*, in preparation.

[22] A. Manning and H. McCluskey, *Hausdorff dimension for horseshoes*, Ergodic Theory Dynamical Systems **3** (1983), 251–260.

[23] P. Mattila, *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability*, Cambridge University Press, Cambridge, U.K., 1995.

[24] M. Misiurewicz, *Rotation Theory*, Misiurewicz's webpage.
[25] S. Newhouse, *Continuity properties of entropy*, Ann. of Math. (2) **129** (1989), 215–235.
[26] Y. Pesin, *Dimension Theory in Dynamical Systems: Contemporary Views and Applications*, Chicago Lectures in Mathematics, Chicago University Press, Chicago, 1997.
[27] Ya. Pesin and B. Pitskel, *Topological pressure and the variational principle for non-compact sets*, (Russian) Funktsional. Anal. i Prilozhen. **18** (1984), 50–63.
[28] H. Poincaré, *Sur les cousbes définies par les équations différentielles*, Euvres Complètes, tome 1, Gauthier-Villars, Paris, (1952), 137-158.
[29] F. Przytycki and M. Urbanski, *Conformal fractals: ergodic theory methods*, London Mathematical Society Lecture Note Series, 371. Cambridge University Press, Cambridge, 2010. x+354 pp.
[30] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.
[31] D. Ruelle, *Repellers for Real Analytic Maps*, Ergodic Theory and Dynamical Systems **2** (1982), 99–107.
[32] D. Ruelle, *Thermodynamic Formalism*, Cambridge: Cambridge Univ. Press, 2004.
[33] F. Takens and E. Verbitskiy, *On the variational principle for the topological entropy of certain non-compact sets*, Ergodic Theory Dynam. Systems **23** (2003), 317-348.
[34] D. Thompson, *A variational principle for topological pressure for certain non-compact sets*, Journal of the London Mathematical Society **81** (2009), 585-602.
[35] D. Thompson, *A thermodynamic definition of topological pressure for non-compact sets*, Ergodic Theory and Dynamical Systems **31** (2011), 527–547.
[36] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics 79, Springer, 1981.
[37] K. Ziemian, *Rotation sets for subshifts of finite type*, Fund. Math. **146** (1995), 189–201.

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