ENTROPY AND ROTATION SETS: A TOYMODEL APPROACH

TAMARA KUCHERENKO AND CHRISTIAN WOLF

Abstract. Given a continuous dynamical system f on a compact metric space X and a continuous potential $\Phi : X \to \mathbb{R}^m$, the generalized rotation set is the subset of $\mathbb{R}^m$ consisting of all integrals of Φ with respect to all invariant probability measures. The localized entropy at a point in the rotation set is defined as the supremum of the measure-theoretic entropies over all invariant measures whose integrals produce that point. In this paper we provide an introduction to the theory of rotation sets and localized entropies. Moreover, we consider a shift map and construct a Lipschitz continuous potential, for which we are able to explicitly compute the geometric shape of the rotation set and its boundary measures. We show that at an exposed point on the boundary there are exactly two ergodic localized measures of maximal entropy.

1. Motivation

An important goal in dynamical systems is to understand the various typical dynamical behaviors of a given system. By Birkhoff's Ergodic Theorem we can associate typical dynamical behavior with an invariant ergodic probability measure. However, for many systems the set of invariant measures is rather large. This raises the question as to which invariant measure is the natural choice to consider.

One natural candidate to consider is a measure that maximizes a certain topological complexity, i.e. entropy, among all invariant probability measures. Such a measure (if it exists) is called a measure of maximal entropy. On the other hand, many systems exhibit an abundance of invariant measures in which case the restriction to one invariant measure may result in the loss of other relevant dynamical information associated with other measures. A natural approach to overcome this problem is to partition the space of invariant measures into smaller classes and then to consider entropy maximizing measures within these classes. In this paper we partition the space of invariant measures by identifying measures according to their integral averages on a given predescribed set of continuous potentials. While

Key words and phrases. topological entropy, generalized rotation sets, variational principle, entropy maximizing measures, localized thermodynamic formalism.

This work was partially supported by grants from the PSC-CUNY (TRADA-45-278 to Tamara Kucherenko), (TRADA-45-356 to Christian Wolf) and by a grant from the Simons Foundation (#209846 to Christian Wolf).

increasing the size of the set of potentials we obtain a decreasing sequence of partition elements and are able to analyze how maximal entropy varies among the invariant probabilities.

To make this precise, let us consider a discrete-time dynamical system given by a continuous map $f : X \to X$ on a compact metric space (X, d). Given a finite set of continuous potentials $\phi_1, \cdots, \phi_m : X \to \mathbb{R}$ we write $\Phi = (\phi_1, \cdots, \phi_m)$. For an f-invariant probability measure μ on X we call

$$\mathrm{rv}_\Phi(\mu) = \left(\int \phi_1 d\mu, \cdots, \int \phi_n d\mu \right)$$

the rotation vector of μ with respect to Φ. Following [15], [12] and [20] we call the set of rotation vectors of all f-invariant probability measures the (generalized) rotation set of f with respect to the m-dimensional potential Φ (see Section 2.1 for details and further references). Given a point w in the rotation set, the rotation class of w is the set of all measures whose rotation vectors produce w.

Rotation classes can be used to partition the space of invariant measures. This approach is summarized as follows. Consider a sequence $(\phi_k)_{k\in\mathbb{N}}$ of continuous potentials that is dense in the Banach space $C(X, \mathbb{R})$ endowed with the supremum norm. Let $\mathrm{Rot}(n)$ denote the rotation set of the potential $\Phi_n = (\phi_1, \cdots, \phi_n)$. It follows that

$$\mathrm{Rot}(n+1) = \bigcup_{w\in\mathrm{Rot}(n)} w \times I_w,$$

where I_w is a compact interval defined by

$$I_w = \left\{ \int \phi_{n+1} d\mu : \ \mu \text{ is } f\text{-invariant and } \mathrm{rv}_{\Phi_n}(\mu) = w \right\}.$$

We say a sequence $(w_n)_{w_n\in\mathrm{Rot}(n)}$ is decreasing if $w_{n+1} = w_n \times \{\alpha_n\}$ for some $\alpha_n \in I_{w_n}$ and all $n \in \mathbb{N}$. It follows that every decreasing sequence $(w_n)_n$ is associated with a decreasing sequence of rotation classes. Moreover, by the Riesz Representation theorem the intersection of these rotation classes contains precisely one invariant measure μ_∞. We say $(\mathrm{Rot}(n))_n$ is a filtration of the space of invariant probability measures. Thus, if n is large we can consider $\mathrm{Rot}(n)$ as a fairly good approximation of of the space of invariant probability measures.

Let now Φ be a general m-dimensional continuous potential and let w be a point in the rotation set of Φ. Following [15] and [20] we call

$$h_{\mathrm{m}}(w) = \sup\{h_\mu(f) : \mathrm{rv}_\Phi(\mu) = w\} \tag{1}$$

the localized entropy at w with respect to Φ (see Section 2.2 for details and references). Here $h_\mu(f)$ denotes the metric entropy of f with respect to μ. Moreover, we say μ is a localized measure of maximal entropy at w if $h_\mu(f) = h_{\mathrm{m}}(w)$ and $\mathrm{rv}_\Phi(\mu) = w$. There are several natural questions that arise within this context:

1. Can we compute the shape of the rotation set for a given potential? Which shapes can be realized as rotation sets within a class dynamical systems?
2. What is the regularity of the map $w \mapsto h_{\mathrm{m}}(w)$?
3. For which systems and potentials do there exist localized measures of maximal entropy? Under which assumptions are these measures unique? Which properties do localized measures of maximal entropy possess?

It turns out that only limited answers to these questions are available in the literature. Even the problem to determine the shape of the rotation set can be extremely difficult when we consider general classes of systems and potentials. Our goal in this paper is twofold. In Section 2 we provide an overview about what is known for rotation sets, localized entropy and localized topological pressure with the particular focus on the questions (1)-(3). Finally, in Section 3 we introduce a toy model of a dynamical system together with a 2-dimensional potential for which we are able to answer questions (1)-(3) completely. While being fairly simple, this model also displays certain phenomena that have not been previously observed.

2. Historical background.

2.1. **Rotation Theory.** The terminology and ideas behind the rotation theory come from the classical example of a continuous map on a torus. Let $\mathbb{T}^m$ denote the m-dimensional real torus, $f: \mathbb{T}^m \to \mathbb{T}^m$ be a continuous map homotopic to the identity and F be any lifting of f to the covering space $\mathbb{R}^m$. The aim of the rotation set is to measure the average movement of any point under iteration of the map. Thus, we are interested in the vectors $\frac{F^n(x)-x}{n}$ for $x \in \mathbb{R}^m$ and large n. The pointwise rotation set of F is defined as a set of limits of statistical averages of the form

$$\left(\frac{F^{n_l}(x_l) - x_l}{n_l}\right)_{l=1}^{\infty}, \quad x_l \in \mathbb{R}^m,\ n_l \to \infty. \tag{2}$$

Note that here we also may vary the point x. This definition of the rotation set was introduced by Misiurewicz and Ziemian [26] in 1989 as a generalization of the Poincaré's rotation number of an orientation preserving homeomorphism on a circle. The celebrated theorem of Poincaré [29] states that if f is an orientation preserving homeomorphism on a circle (m=1) then all limits in (2) do not depend on x and produce only one point, which is the rotation number of the homeomorphism.

We can also describe rotation sets in terms of the displacement function $\Phi: \mathbb{T}^m \to \mathbb{R}^m$ defined as $\Phi(\pi(x)) = F(x) - x$, where $\pi: \mathbb{R}^m \to \mathbb{T}^m$ denotes the natural projection. Then statistical averages in (2) can be written as a telescopic sum

$$\frac{F^n(x) - x}{n} = \frac{1}{n}\sum_{k=0}^{n-1} \Phi(f^k(y)), \quad \text{where}\, y = \pi(x).$$

Therefore, studying rotation sets is closely related to studying limits of statistical averages of the displacement function. The Birkhoff's Ergodic Theorem asserts that for an ergodic f-invariant probability measure μ we have that

$$\frac{1}{n}\sum_{i=0}^{n-1}\Phi(f^i(y)) \to \int \Phi\, d\mu \quad \mu - a.e. \tag{3}$$

Therefore, the set of all integrals of Φ with respect to ergodic f-invariant measures is contained in the pointwise rotation set. What happens when we drop the ergodicity restriction and consider all f-invariant probability measures? Misiurevicz and Ziemian proved that in this case the set of all integrals of the displacement function Φ with respect to f-invariant probability measures is exactly the convex hull of the pointwise rotation set of F [25]. The set of such integrals was also introduced by Herman in [14] as an alternative definition of the rotation set. We adopt this definition.

Definition. *The (classical) rotation set of a displacement function* Φ *is defined by*

$$\mathrm{Rot}(\Phi) = \left\{\int \Phi\, d\mu : \mu \textit{ is an } f\textit{-invariant probability measure}\right\}.$$

The word classical here refers to the fact that Φ is a displacement function of some torus homeomorphism. Since the set of invariant probability measures is convex and weak* compact, the rotation set is always a compact convex subset of $\mathbb{R}^m$. This raises the natural question which compact convex subsets of $\mathbb{R}^m$ can be realized as rotation sets of a displacement function of a torus homeomorphism. This problem is still open after more than 20 years. We refer to [22, 23] and [8] for partial results.

There is a natural transition from the rotation theory on the torus to the abstract situation where we have a general dynamical system and an arbitrary continuous potential instead of a displacement function. The theory of generalized rotation sets was largely developed by Block [3], Geller and Misiurewicz [12], Ziemian [32] and Jenkinson [15].

Suppose X is an arbitrary compact metric space and $f : X \to X$ is continuous. Denote by $\mathcal{M}$ the set of all f-invariant Borel probability measures on X endowed with the weak* topology. This makes $\mathcal{M}$ a compact convex topological space. Let $\mathcal{M}_E \subset \mathcal{M}$ be a subset of ergodic measures. Consider a continuous m-dimensional potential $\Phi = (\phi_1, \cdots, \phi_m) : X \to \mathbb{R}^m$ and let $\mathrm{rv}_\Phi(\mu) = \left(\int \phi_1\, d\mu, \dots, \int \phi_m\, d\mu\right)$ be the rotation vector of the measure μ. There are three ways to define a rotation set of Φ.

Definition 1. *The (generalized) rotation set of* Φ *with respect to* f *is*

$$\mathrm{Rot}(\Phi) = \{\mathrm{rv}_\Phi(\mu) \,:\, \mu \in \mathcal{M}\}. \tag{4}$$

Definition 2. *The ergodic rotation set of* Φ *with respect to* f *is*

$$\mathrm{Rot_E}(\Phi) = \{\mathrm{rv}_\Phi(\mu) \,:\, \mu \in \mathcal{M}_\mathrm{E}\}. \tag{5}$$

Definition 3. *The pointwice rotation set of* Φ *with respect to* f *is*

$$\mathrm{Rot}_{\mathrm{Pt}}(\Phi) = \left\{ \lim_{l\to\infty} \frac{1}{n_l} \sum_{i=1}^{n_l} \Phi(f^i(x_l)) : (x_l) \subset X,\, n_l \to \infty \right\}. \tag{6}$$

We have the following inclusions.

$$\mathrm{Rot}_{\mathrm{E}}(\Phi) \subset \mathrm{Rot}_{\mathrm{Pt}}(\Phi) \subset \mathrm{Rot}(\Phi). \tag{7}$$

The left-hand side inclusion follows from the Birkhoff Ergodic theorem. The right-hand side inclusion is a consequence of the Banach-Alaoglu theorem and sequential compactness of the set of Borel probability measures. Indeed, consider a sequence of measures which are supported on the points of corresponding ergodic averages in (6). In general, these measures are not f-invariant. However, any accumulation point is, and the integral over this measure coincides with the statistical limit.

In [20] we study the relationship between the pointwise, ergodic and generalized rotation sets of Φ. In particular, we construct explicit examples showing that both of the inclusions in (7) can be strict. Our construction utilizes the fact that $\mathrm{Rot}_{\mathrm{E}}(\Phi)$ and $\mathrm{Rot}_{\mathrm{Pt}}(\Phi)$ are not necessarily convex, but $\mathrm{Rot}(\Phi)$ is. On the other hand, we can extend the result in [25] to the abstract setting and show that the generalized rotation set is the convex hull of the pointwise (or ergodic) rotation set.

Proposition. *Let* $f: X \to X$ *and* $\Phi : X \to \mathbb{R}^m$ *be continuous maps on a compact metric space* X*. Then*

$$\mathrm{Conv}\,\mathrm{Rot}_{\mathrm{E}}\Phi = \mathrm{Conv}\,\mathrm{Rot}_{\mathrm{Pt}}(\Phi) = \mathrm{Rot}(\Phi).$$

Proof. In view of (7) it suffices to show that the extreme points of $\mathrm{Rot}(\Phi)$ are rotation vectors of ergodic measures. Let w be an extreme point of $\mathrm{Rot}(\Phi)$ and let $\mu \in \mathcal{M}$ be such that $\mathrm{rv}_\Phi(\mu) = w$. Consider an ergodic decomposition τ of μ, that is, τ is a Borel probability measure on $\mathcal{M}$ with $\tau(\mathcal{M}_E) = 1$ and

$$\int_X \varphi\, d\mu = \int_{\mathcal{M}} \int_X \varphi\, d\nu(x)\, d\tau(\nu)$$

holds for all $\varphi \in C(X,\mathbb{R})$. It follows that

$$w = \int \Phi\, d\mu = \int_{\mathcal{M}} \int_X \Phi\, d\nu(x)\, d\tau(\nu) = \int_{\mathcal{M}} \mathrm{rv}_\Phi(\nu)\, d\tau(\nu).$$

If $\nu \in \mathcal{M}_E$ then $\mathrm{rv}_\Phi(\nu) \in \mathrm{Rot}_{\mathrm{E}}(\Phi) \subset \mathrm{Rot}(\Phi)$. Using that w is an extreme point of $\mathrm{Rot}(\Phi)$ we conclude that $\mathrm{rv}_\Phi(\nu) = w$ for τ-almost all $\nu \in \mathcal{M}$. It follows that $w \in \mathrm{Rot}_{\mathrm{E}}(\Phi)$. □

For various dynamical systems the pointwise rotation set and generalized rotation set coincide. The reason is as follows: Suppose $w \in \mathrm{Rot}(\Phi)$. If any neighborhood of w contains a rotation vector of an ergodic measure then $w \in \mathrm{Rot}_{\mathrm{Pt}}(\Phi)$. This is in particular true if the ergodic invariant measures are weak* dense in the set of all invariant measures in which case the pointwise and general rotation sets are one and the same. For example, these sets coincide for topologically mixing subshifts of finite type, expansive homeomorphisms with specification, axiom A basic sets. On the other hand, the inclusion $\mathrm{Rot}_{\mathrm{E}}(\Phi) \subset \mathrm{Rot}_{\mathrm{Pt}}(\Phi)$ maybe strict even for shift maps (see [20]).

Considerable attention has been given to the question of the possible geometry of the generalized rotation sets for various classes of dynamical systems. It follows from the convexity and weak* compactness of the set of invariant probability measures that the rotation set is always a compact convex subset of $\mathbb{R}^m$. The boundary of a convex compact set is Lipschitz and therefore almost everywhere differentiable. It is tempting to believe that at least for systems with "nice" properties (for example Axiom A and Hölder potentials) the boundary of the rotation set is piecewise smooth. However, this is in general not true. A counter-example was given by Bousch (see [5, 16]). It is shown that if $f(z) = z^2$ on S^1 and $\Phi = id_{S^1}$ then the boundary of the corresponding rotation set is non-differentiable on a countable dense set.

One of the earliest results regarding the geometry of rotation sets is due to Ziemian [32]. She studied the case of transitive sub-shifts of finite type and the potential being constant on cylinders of size two, i.e. it depends only on the first and second coordinates. She showed that under these assumptions the rotation set is a polyhedron.

In [22] Kwapisz considers the opposite question: which subsets of $\mathbb{R}^2$ can arise as rotation sets for torus homeomorphisms. He proved that any polygon whose vertices are at rational points in the plane can be obtained as the rotation set of some homeomorphism on the two-torus. Years later, Passeggi [28] proved that maps whose rotation sets are rational polygons form an open and dense set among all torus homeomorphisms homotopic to the identity. Moreover, the rotation set of any axiom A diffeomorphism on a torus is a rational polygon.

In general, rotation sets of torus homeomorphisms need not be polygons. Namely, Kwapisz constructed an explicit example of a rotation set that is a "polygon" with infinitely many vertices [23]. He also proved that certain line segments on the plane cannot be realized as a rotation set of a torus homeomorphism [24].

In a recent paper [8], Boyland, Carvalho and Hall provide new examples of homeomorphisms on a torus with rotation sets being infinite polygons. In particular, they construct an example of a rotation set that has infinitely many rational polygonal vertices accumulating on a single smooth extreme point. However, it is still not known whether the rotation set of a torus homeomorphism can be a strictly convex set.

In [20] we consider subshifts of finite type and show that in contrast to the case of torus homeomorphisms, there are no restrictions on the geometry of a rotation set. We prove that every compact convex set is attained as a rotation set of a continuous potential.

Theorem. [20] *Let K be a compact convex subset of $\mathbb{R}^m$. Then there exist a shift map f on a shift space Σ with finite alphabet and a continuous potential $\Phi : \Sigma \to \mathbb{R}^m$ such that* $\mathrm{Rot}(\Phi) = K$.

The proof of this theorem is rather technical. Therefore, we outline here only the general ideas and refer to [20] for the details. The proof is based on an approximation argument. For simplicity, we consider here only the 2-dimensional case and a full one-sided shift with alphabet {0,1}. Let K be any compact convex set in $\mathbb{R}^2$. The idea is to approximate K by polygons $\mathcal{P}_n$ with vertices on the boundary of K and construct a sequence of continuous potentials Φ_n with $\mathrm{Rot}(\Phi_n) = \mathcal{P}_n$ which converges uniformly to a potential Φ. Then, $\mathrm{Rot}(\Phi) = K$. The challenge, however, is that it is not possible to achieve both, $\mathrm{Rot}(\Phi_n) = \mathcal{P}_n$ and uniform convergence. It is sufficient instead, to have $\mathrm{Rot}(\Phi_n) \approx \mathcal{P}_n$.

In general, to get a potential whose rotation set is a polygon we partition the shift space into cylinders and define the values of the potential on these cylinders to be the vertices. This only guaranties that the rotation set is inside the given polygon. To obtain exactly the desired polygon we need rotation vectors that are vertices. It suffices to have a periodic orbit for each vertex which is mapped into that vertex by the potential. Since a periodic orbit intersects several cylinders, the potential should have the same value on all of them. We need to cluster the cylinders according to our selected periodic orbits and define the values of the potential to be the corresponding vertex on each cluster. This is the general idea, but we need to assure that the obtained sequence of potentials converges uniformly.

To pass from the potential Φ_n to the next potential Φ_{n+1} in the sequence, we increase the number of vertices of the polygon $\mathcal{P}_n$. We pick a periodic orbit and define Φ_{n+1} so that the rotation vector of this orbit is one of the new vertices. However, note that the values of the previous potential Φ_n on this orbit might be the points on the other side of K and this situation would not allow for uniform convergence. To solve this problem we pick an orbit in such a way that its Φ_n values hit the neighborhood of the selected vertex much more often then any other. Then we let Φ_{n+1} be the same as Φ_n in far away points. We do not get the new vertex exactly as a rotation vector of Φ_{n+1}, but a point reasonably close. Iteratively applying these steps we are able to construct a sequence of potentials Φ_n with the desired properties.

2.2. **Localized entropy.** One of the reasons that inspired the study of rotation sets is the connection between arithmetic properties of rotation numbers and dynamical properties of the circle maps. For each rational number from the rotation interval of a continuous circle map of degree one there is a periodic point with such rotation number [4]. A generalization of this result to arbitrary tori was established by Franks [11]. Thus, the rotation interval provides some information about the periodic orbits of the map, which could be used in particular to study the entropy. The topological entropy characterizes the complexity of the map. Roughly speaking, entropy measures the exponential growth rate of the number of different orbits as we increase the length of the orbits. Frequently, positive entropy is used as a notion for a system being chaotic.

There are several theorems about the structure of the rotation set of a map that also provide entropy estimates. In the case of continuous circle maps of degree one (not necessarily orientation preserving) the sharp estimate on the minimal topological entropy of a map with a given rotation interval was established in [1]. The topological entropy is at least the logarithm of the largest real root of a polynomial whose coefficients are derived from the end-points of the rotation interval. Later the lower and upper bounds of the topological entropy and of the number of periodic orbits of each period were obtained for circle maps of degree one with a single maximum and a single minimum (see [2]).

We now give the definition of the topological entropy for a continuous map f on a compact metric space (X, d). Here, we follow the approach of Bowen and Dinaburg. The topological entropy of f measures the exponential growth rate of distinguishable orbits as we increase the iteration of the map f. To distinguish the orbits we define a new metric on X. For $x, y \in X$ and $n \in \mathbb{N}$ the distance between x and y is defined by

$$d_n(x,y) = \max_{k=0,\dots,n-1} d(f^k(x), f^k(y)).$$

Hence, $d_n(x, y)$ is the maximal distance the points spread apart after $n - 1$ iterations of the map. Note that d_n is a metric (called the Bowen metric) that induces the same topology on X as d.

For $\varepsilon > 0$ and $n \in \mathbb{N}$ we say that $F \subset X$ is (n, ε)-separated if $d_n(x, y) \geq \varepsilon$ for any $x, y \in F$, $x \neq y$. Then F is a set of points we can distinguish after $n - 1$ iterations of the map f with a measurement having precision ε. Let $F_n(\varepsilon)$ denote a maximal (with respect to the inclusion) (n, ε)-separated set. The *topological entropy* of f is defined by

$$h_{\rm top}(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \operatorname{card} F_n(\varepsilon).$$

Although the sets $F_n(\varepsilon)$ appear explicitly in the definition above, the topological entropy is in fact independent on the choice of a sequence of maximal (n, ε)-separated sets [31].

Given an f-invariant measure μ we can associate the measure-theoretic entropy of f with respect to μ, denoted by $h_\mu(f)$ (see [31] for the definition and details). Roughly speaking, $h_\mu(f)$ measures the complexity of the system by ignoring sets of μ-measure zero. Intuitively, measures assigning most weight to the regions of high complexity should have measure-theoretic entropy close to the topological entropy. This statement is made precise by the variational principle for the entropy, namely

$$\mathrm{h}_{\rm top}(f) = \sup\{\mathrm{h}_\mu(f) : \mu \in \mathcal{M}\}. \tag{8}$$

A measure at which this supremum is attained is called a *measure of maximal entropy*. The study of entropy maximizing measures (existence, uniqueness and properties) has a long history and the results are widely spread in the literature [6, 9, 13, 17], yet a complete understanding is still lacking today.

For each point w in a rotation set we can associate local versions of the measure-theoretic and the topological entropies. In the case of a torus and the displacement function, localization means that we are only interested in the entropy of the points which on average move in the direction of w. The localized measure-theoretic entropy in the general context (arbitrary systems and potentials) has been extensively studied by Geller and Misiurewicz in [12] as well as by Jenkinson in [15]. It is defined as the supremum of the entropies of the measures whose integrals of Φ produce the vector w. Precisely, for $w \in \mathrm{Rot}(\Phi)$, we define the *localized measure-theoretic entropy* at w (with respect to Φ and f) by

$$h_{\mathrm{m}}(w, f, \Phi) = h_{\mathrm{m}}(w) = \sup \{h_\mu(f) : \mu \in \mathcal{M}_\Phi(w)\}, \tag{9}$$

where $\mathcal{M}_\Phi(w) = \{\mu \in \mathcal{M} : \mathrm{rv}_\Phi(\mu) = w\}$ is the *rotation class* of w. A measure at which the supremum is attained is called the *localized measures of maximal entropy at* w. The notion of a localized measure of maximal entropy is analogous to a classical measure of maximal entropy with the exception that we here only consider invariant measures in $\mathcal{M}_\Phi(w)$ rather than all invariant measures. For systems with an upper semi-continuous entropy map there exists at least one localized measure of maximal entropy. Unlike in the classical case, there does not need to exist an ergodic localized measure of maximal entropy (see [21, Ex. 3]).

The properties of the localized measures of maximal entropy strongly depend on the geometric location of w in the rotation set. For example, for subshifts of finite type and Hölder continuous potentials, if w is in the (relative) interior of the rotation set, then there exists a unique measure of maximal entropy at w. However, at the boundary points of the rotation set measures of maximal entropy may not be unique. Indeed, we recently constructed a Lipschitz continuous potential for a shift map that has infinitely many ergodic measures of maximal entropy at a boundary point of its rotation set [21, Thm 2]. In our construction the potential is one-dimensional and its rotation set is an interval. Jenkinson [15] provided several examples of 2-dimensional potentials on subshift spaces whose rotation sets exhibit two ergodic localized measures of maximal entropy at a boundary point. However, in all these examples the boundary point is non-exposed. In higher dimensions the properties of measures of maximal entropy depend on the type of boundary point. Jenkinson's paper raises the question whether an entropy maximizing measure at an exposed point is unique. To the best of our knowlege the example presented in Section 3 is the first one to answer this question: such measures are not necessarily unique.

Alternatively, we may adopt the Bowen-Dinaburg approach to define localized entropy at a vector $w \in \mathrm{Rot}_{\mathrm{Pt}}(\Phi)$ in purely topological terms. It is computed using only orbits whose statistical averages with respect to a given m-dimensional potential Φ are close to w (see [20]).

For $x \in X$ and $n \in \mathbb{N}$, we denote by $\frac{1}{n}S_n(\Phi, f)(x)$ the m-dimensional Birkhoff average at x of length n with respect to Φ and f, where $S_n(\Phi, f)(x) = (S_n(\phi_1, f)(x), \dots, S_n(\phi_m, f)(x))$ and

$$S_n(\phi_i, f)(x) = \sum_{k=0}^{n-1} \phi_i(f^k(x)). \tag{10}$$

Given $w \in \mathbb{R}^m$ and $r > 0$ we say a set $F \subset X$ is a (n, ε, r, w)-set for Φ and f if F is (n, ε)-separated set and for all $x \in F$ the Birkhoff average $\frac{1}{n}S_n(\Phi, f)(x)$ is contained in the open Euclidean ball $B(w, r)$ with center w and radius r. For all $n \in \mathbb{N}$ and $\varepsilon, r > 0$ we pick a maximal (with respect to the inclusion) (n, ε, r, w)-set $F_n(\varepsilon, r, w)$. Then the *localized topological entropy* at $w \in \mathbb{R}^m$ (with respect to Φ and f) is defined by

$$h_{\rm top}(w, \Phi, f) = h_{\rm top}(w) = \lim_{r\to 0} \lim_{\varepsilon \to 0} \limsup_{n\to\infty} \frac{1}{n} \log {\rm card}\ F_n(\varepsilon, r, w). \tag{11}$$

As in the case of the classical topological entropy, this definition is independent of the choice of the sets $F_n(\varepsilon, r, w)$.

Note that the definition of $h_{\rm top}(w, f, \Phi)$ is only meaningful if $B(w, r)$ contains statistical averages for infinitely many n and arbitrarily small r. The set of points in $\mathbb{R}^m$ which satisfy this property is precisely ${\rm Rot}_{\rm Pt}(\Phi)$.

For a wide variety of dynamical systems, including subshifts of finite type, the localized measure-theoretic and topological entropies coincide at every point of the rotation set. This is a consequence of a localized variational principle for the entropy (see [19, 20, 21]).

Localized Variational Principle for Entropy. *Let $f : X \to X$ be a continuous map on a compact metric space X. Let $\Phi : X \to \mathbb{R}^m$ be continuous and let $w \in {\rm Rot}(\Phi)$ be such that the map $v \mapsto h_{\rm m}(v, f, \Phi)$ is continuous at w and $h_{\rm m}(w, f, \Phi)$ is approximated by ergodic measures. Then $h_{\rm top}(w, f, \Phi) = h_{\rm m}(w, f, \Phi)$.*

Here, we say that $h_{\rm m}(w, f, \Phi)$ is *approximated by ergodic measures* if there exists a sequence of ergodic measurs $(\mu_n)_{n\in\mathbb{N}}$ such that ${\rm rv}_\Phi(\mu_n) \to w$ and $h_{\mu_n}(f) \to h_{\rm m}(w, f, \Phi)$ as $n \to \infty$. Note that in this case $w \in {\rm Rot}_{\rm Pt}(\Phi)$. The assumption that $h_{\rm m}(w, f, \Phi)$ is approximated by ergodic measures cannot be dropped in the previous theorem. Indeed, there are examples which do not satisfy this assumption and $h_{\rm top}(w, f, \Phi) < h_{\rm m}(w, f, \Phi)$ holds. On the other hand, without the assumption that $v \mapsto h_{\rm m}(v, f, \Phi)$ is continuous at w, we arrive at the opposite strict inequality [21].

We remark that the continuity of $v \mapsto h_{\rm m}(v, f, \Phi)$ holds at all points w if the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous. In particular, this is true if f is expansive [7], a C^∞ map on a compact smooth Riemannian manifold [27], or satisfies the entropy-expansiveness [10].

2.3. **The localized thermodynamic formalism.** Although the prime focus of this note are the topological and measure-theoretic entropies associated to the points in the rotation set, this theory has been recently extended to the localized topological pressure (see [21]). For completeness we provide here a brief overview. Let $f : X \to X$ be a continuous map on a compact metric space (X, d). Recall that the *topological pressure* (with respect to f) is a mapping $P_{\rm top}\colon C(X,\mathbb{R}) \to \mathbb{R}\cup\{\infty\}$ defined by

$$P_{\rm top}(\varphi) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N_\varphi(n,\varepsilon), \tag{12}$$

where

$$N_\varphi(n,\varepsilon) = \sup\left\{ \sum_{x\in F} e^{S_n\varphi(x)} : F \text{ is } (n,\varepsilon)\text{-separated set} \right\}$$

and $S_n\varphi(x)$ is defined as in (10). It follows from the definitions that the topological entropy of f is $h_{\rm top}(f) = P_{\rm top}(0)$. The topological pressure satisfies the well-known variational principle, namely,

$$P_{\rm top}(\varphi) = \sup_{\mu\in\mathcal{M}} \left(h_\mu(f) + \int_X \varphi\, d\mu \right). \tag{13}$$

It is easy to see that the supremum in (13) can be replaced by the supremum taken only over all $\mu \in \mathcal{M}_{\rm E}$.

If there exists a measure $\mu \in \mathcal{M}$ at which the supremum in (13) is attained it is called an *equilibrium state* (or also equilibrium measure) of the potential φ. In general, the set of all equilibrium states of φ may be empty. Note that if the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous on $\mathcal{M}$ then for each $\varphi \in C(X,\mathbb{R})$ there exists at least one equilibrium state. Since the set of all equilibrium states of φ is a compact and convex subset of $\mathcal{M}$ whose extreme points are the ergodic measures (see [31]), we conclude that in this case there must be at least one ergodic equilibrium state.

To define the localized versions of the pressure we consider continuous potentials $\varphi : X \to \mathbb{R}$ and $\Phi = (\phi_1, \cdots, \phi_m) : X \to \mathbb{R}^m$. We think of φ as our target potential for computing the localized pressure and of Φ as the potential providing the localization.

The localized topological pressure of a continuous potential φ is computed by considering only those (n,ε)-separated sets whose statistical sums with respect to an m-dimensional potential Φ are "close" to a given value $w \in \mathbb{R}^m$. Precisely, given $w \in {\rm Rot}_{\rm Pt}(\Phi)$ and $r > 0$ we say a set $F \subset X$ is an (n,ε,w,r)-set if F is (n,ε)-separated set and for all $x \in F$ the Birkhoff average $\frac{1}{n}S_n(\Phi,f)(x)$ is contained in the Euclidean ball $B(w,r)$ with center w and radius r. We define the *localized topological pressure* of the potential φ (with respect to Φ and w) by

$$P_{\rm top}(\varphi,\Phi,w) = \lim_{r\to 0} \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N_\varphi(n,\varepsilon,w,r), \tag{14}$$

where

$$N_\varphi(n,\varepsilon,w,r) = \sup\left\{\sum_{x\in F} e^{S_n(\varphi,f)(x)} : F \text{ is } (n,\varepsilon,w,r)\text{-set}\right\}.$$

For $w \in \mathrm{Rot}(\Phi)$, we define the *localized measure-theoretic pressure* of the potential φ (with respect to Φ and w) by

$$P_{\mathrm{m}}(\varphi,\Phi,w) = \sup\left\{h_\mu(f) + \int_X \varphi\, d\mu :\ \mu \in \mathcal{M}_\Phi(w)\right\}.$$

In case we take the supremum in the above definition over all invariant measures we obtain the classical measure-theoretic pressure. In contrast to the classical situation where the topological and measure-theoretic pressure concide (see (13)), it turns out that in the case of localized pressure the measure-theoretic and topological pressures may differ, and strict inequalities can occur in both directions. On the other hand, the following result gives a fairly complete description of the assumptions needed to still have a variational principle.

Localized Variational Principle for Topological Pressure. [21] *Let $f : X \to X$ be a continuous map on a compact metric space X. Let $\varphi : X \to \mathbb{R}$ and $\Phi : X \to \mathbb{R}^m$ be continuous and let $w \in \mathrm{Rot}(\Phi)$ be such that the map $v \mapsto P_m(\varphi,\Phi,v)$ is continuous at w and $P_{\mathrm{m}}(\varphi,\Phi,w)$ is approximated by ergodic measures. Then*

$$P_{\mathrm{top}}(\varphi,\Phi,w) = P_{\mathrm{m}}(\varphi,\Phi,w).$$

We note that Theorem 2.3 holds for a wide variety of systems and potentials such as expansive homeomorphisms with specification, which include topological mixing two-sided subshifts of finite type as well as diffeomorphisms with a locally maximal topological mixing hyperbolic set.

Let us focus on equilibrium states that arise from the localized variational principle. We fix $w \in \mathrm{Rot}(\Phi)$ and consider its rotation class $\mathcal{M}_\Phi(w) = \{\mu \in \mathcal{M} : \mathrm{rv}_\Phi(\mu) = w\}$. We say $\mu \in \mathcal{M}_\Phi(w)$ is a *localized equilibrium state* of $\varphi \in C(X,\mathbb{R})$ (with respect to Φ and w) if

$$h_\mu(f) + \int_X \varphi\ d\mu = \sup_{\nu\in\mathcal{M}_\Phi(w)} \left(h_\nu(f) + \int_X \varphi\, d\nu\right). \tag{15}$$

This definition is analogous to that of a classical equilibrium state with the exception that we here only consider invariant measures in $\mathcal{M}_\Phi(w)$ rather than all invariant measures. If the localized variational principle holds at w then (15) is equivalent to

$$P_{\mathrm{top}}(\varphi,\Phi,w) = h_\mu(f) + \int_X \varphi\ d\mu.$$

One class of systems exhibiting the strongest possible properties for classical equilibrium states are the expansive homeomorphisms with specification which include topological mixing subshifts of finite type as well as diffeomorphisms with a locally maximal topological mixing hyperbolic set. For such systems, given a Hölder continuous potential φ, there exists a unique equilibrium state μ_φ (which is ergodic) and μ_φ has the Gibbs property (see [6, 13, 18, 30]). This result does not carry over to localized equilibrium states [21]. The example presented in Section 3 exhibits exactly two ergodic localized equilibrium states associated to a boundary point of a rotation set, none of which is Gibbs. In [21, Thm. B] we show that these phenomena do not occur in an interior point of the rotation set.

In case of systems with strong thermodynamic properties and Hölder continuous potentials, the interior equilibrium states still share many of the properties of classical equilibrium states (see [21]). Indeed, if $\dim \mathrm{Rot}(\varphi, \Phi) = m$ then there exists a unique (ergodic) localized equilibrium state at $w \in \mathrm{int}\ \mathrm{Rot}(\Phi)$. This holds in particular for $\varphi \equiv 0$ and more generally if φ is cohomologous to a constant. Therefore, the assumption $w \in \mathrm{int}\ \mathrm{Rot}(\Phi)$ implies the existence of a unique localized measure of maximal entropy. If $\dim \mathrm{Rot}(\varphi, \Phi) = m + 1$ then the set of ergodic localized interior equilibrium states is non-empty and finite. Another interesting property of interior localized equilibrium states is that they are classical equilibrium states of the potentials $\alpha \cdot \Phi + \beta\varphi$ for some $\alpha \in \mathbb{R}^m$ and $\beta \in \mathbb{R}$. This implies that for subshifts of finite type, uniformly hyperbolic systems or expansive homeomorphisms with specification, any ergodic localized equilibrium state of a Hölder continuous potential in the interior of the rotation set is a Gibbs state.

3. A toy model example

In this section we consider a shift map and construct a Lipschitz continuous potential Φ, for which we are able to explicitly describe the rather complex shape of the rotation set and its boundary measures. We call our example of the corresponding rotation set the "fish" due its shape (see Figure 1). We show that at an exposed point w on the boundary of the rotation set there are exactly two ergodic measures of maximal entropy. To prove that the measures in question are indeed maximal we explicitly compute the topological entropy at w and apply the localized variational principle. Moreover, by slightly modifying this example, we show that the cardinality of ergodic localized equilibrium states is in general not preserved under small perturbations of the potential.

We start by introducing some basic concepts about shift maps that will be used later on.

Let $d \in \mathbb{N}$ and let $\mathcal{A} = \{0, \cdots, d-1\}$ be a finite alphabet in d symbols. The (one-sided) shift space X on the alphabet $\mathcal{A}$ is the set of all sequences $\xi = (\xi_n)_{n=1}^\infty$ where $\xi_n \in \mathcal{A}$ for all $n \in \mathbb{N}$. We endow X with the Tychonov product topology which makes X a compact metrizable space. For example, given $0 < \alpha < 1$ it is easy to see that

$$d(\xi,\eta) = d_\alpha(\xi,\eta) = \alpha^{\inf\{n\in\mathbb{N}:\ \xi_n\neq\eta_n\}} \tag{16}$$

defines a metric which induces the Tychonov product topology on X. The shift map $f : X \to X$ (defined by $f(\xi)_n = \xi_{n+1}$) is a continuous d to 1 map on X. If $Y \subset X$ is an f-invariant set we say that $f|_Y$ is a subshift. In particular, for a $d\times d$ matrix A with values in $\{0,1\}$ we define $X_A = \{\xi \in X : A_{\xi_n,\xi_{n+1}} = 1\}$. It is easy to see that X_A is a closed (and therefore compact) f-invariant set and we say that $f|_{X_A}$ is a subshift of finite type. A subshift of finite type is (topologically) mixing if A is aperiodic, that is, if there exists $n \in \mathbb{N}$ such that $A^n_{i,j} > 0$ for all $i,j \in \mathcal{A}$.

Analogously, we obtain the concept of two-sided shift spaces and shift maps by defining X to be the space of all bi-infinite sequences $\xi = (\xi_n)_{n=-\infty}^{\infty}$ where $\xi_n \in \mathcal{A}$ for all $n \in \mathbb{Z}$.

Example 1. *Let $f : X \to X$ be the one-sided full shift with alphabet $\{0,1,2,3\}$. Let C be a compact and convex subset of $\mathbb{R}^2$ whose boundary ∂C is a strictly convex Jordan curve. Pick any point $w_\infty \in \partial C$. Then there exists a line passing through w_∞ which does not intersect $\operatorname{int} C$, but its orthogonal line does. Let w_0 be any point in $\operatorname{int} C$ on that orthogonal line and let v_1, v_2 be points on ∂C on opposite sides with respect to the line. Denote by l_1, l_2 the arcs in ∂C joining v_1, v_2 and w_∞. For $i = 1,2$ we pick a strictly unidirectional sequence $(v_i(k))_{k\in\mathbb{N}} \subset l_i$ starting at v_i and going towards w_∞. We require that $v_i(1) = v_i$ and $|v_i(k) - w_\infty| < 1/2^k$ for all $k > 1$, in particular $\lim_{k\to\infty} v_i(k) = w_\infty$.*

Next, we define several subsets of X. Let $S_1 = \{0,1\}, S_2 = \{2,3\}$ and fix $\alpha \in \mathbb{N}, \alpha \geq 3$. For $i = 1,2$ and all $k \geq \alpha$ we define $Y_i(k) = \{\xi \in X : \xi_1,\dots,\xi_k \in S_i\}$. Moreover, let $Y_0(\alpha) = X \setminus (Y_1(\alpha) \cup Y_2(\alpha))$.

Finally, we define a potential $\Phi : X \to \mathbb{R}^2$ by

$$\Phi(\xi) = \begin{cases} w_0 & \text{if } \xi \in Y_0(\alpha) \\ v_i(k-\alpha) & \text{if } \xi \in Y_i(k-1) \text{ and } \xi \notin Y_i(k),\ k > \alpha \\ w_\infty & \text{if } \ \xi \in Y_i(k) \text{ for all } k \text{ for some } i \in \{1,2\} \end{cases} \tag{17}$$

Note that $\Phi(x) = w_\infty$ if and only if either $\xi_k \in \{0,1\}$ for all $k \in \mathbb{N}$ or $\xi_k \in \{2,3\}$ for all $k \in \mathbb{N}$, in particular $f|_{\Phi^{-1}(w_\infty)}$ is a subshift finite type f_{A} with transition matrix

$$\mathrm{A} = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}. \tag{18}$$

To illustrate this example we consider a case where the set C and the sets of points $v_1(k)$, $v_2(k)$ are symmetric about the line through w_∞ and w_0. We denote by $w_i(j)$ the rotation vectors of the periodic orbits of length j whose generators have the first $j-1$ coordinates in S_i and the j^{th} coordinate in the complementary alphabet S_{3-i}. Precisely, for $j > \alpha$ and $i = 1,2$ we have

$$w_i(j) = \frac{\sum_{k=1}^{j-\alpha} v_i(k) + \alpha w_0}{j}. \tag{19}$$

We show that in this case the boundary of $\mathrm{Rot}(\Phi)$ is the infinite polygon. Moreover, there is a neighborhood of w_∞ where the vertices of $\mathrm{Rot}(\Phi)$ are exactly $w_i(j)$, $i = 1, 2$ and $j > j_0$ for some integer j_0 which depends on properties ∂C. We prove this fact in the next proposition by introducing a new coordinate system with the origin at w_∞ and and the x-axis passing through w_0. For a given point $a \in \mathbb{R}^2$ we write $\mathrm{pr}_x(a)$ and $\mathrm{pr}_y(a)$ for the x and y coordinates respectively. Let $y = l(x)$ denote the parametrized boundary curve of the upper half of C. By symmetry $y = -l(x)$ coincides with the boundary curve of the lower half. For simplicity we add an additional assumption on $l(x)$ that guaranties that $j_0 = \alpha$.

Proposition 1. *Suppose that $\{x_k\} \subset [0,1]$ is a decreasing sequence such that $x_k \leq \frac{1}{2^k}$, $l : [0,1] \mapsto \mathbb{R}$ is an increasing and strictly convex function such that $l(0) = 0$ and $l(x_1) > (\alpha+1)l(x_2)$. Let w_0 be the midpoint between x_1 and x_2. For $i = 1, 2$ denote by $v_i(k) = (x_k, (-1)^i l(x_k))$, let $w_i(j)$ be as in (19) for $j > \alpha$ and set $w_i(\alpha) = \left(\frac{(\alpha-1)\mathrm{pr}_x(w_0)+x_1}{\alpha}, (-1)^i \frac{l(x_1)}{3\alpha}\right)$. Then for the potential Φ defined in Example 1 we have*

$$\mathrm{Rot}(\Phi) = \overline{\mathrm{Conv}}\{w_i(j) : j \geq \alpha,\, i = 1, 2\}. \tag{20}$$

Proof. First we show that the sequence of points $\{w_1(j)\}_{j>\alpha}$ is monotonically decreases to the origin. By symmetry, this immediately implies that the sequence $\{w_2(j)\}_{j>\alpha}$ increases monotonically to the origin. It follows from (19) that for any $j > \alpha$ we have

$$w_1(j) - w_1(j+1) = \frac{1}{j(j+1)}\left[\alpha w_0 + \sum_{k=1}^{j-\alpha} v_1(k) - j v_1(j+1-\alpha)\right]. \tag{21}$$

The x-coordinate of $w_1(j) - w_1(j+1)$ is always positive, since the x_k are decreasing and $\mathrm{pr}_x(w_0) > x_{j+1-\alpha}$. The y-coordinate of $w_1(j) - w_1(j+1)$ simplifies to

$$\sum_{k=1}^{j-\alpha} l(x_k) - j l(x_{j+1-\alpha}). \tag{22}$$

This expression is positive whenever $l(x_1) > (\alpha+1)l(x_{j+1-\alpha})$. This can always be achieved starting from some j_0 since $l(x_k)$ is decreasing to zero. Therefore, $w_1(j)$ are decreasing for $j > j_0$. The assumptions of the proposition assure that we may take $j_0 = \alpha$, however this condition is not essential.

The result by Sigmund that the periodic point measures are dense in $\mathcal{M}$ reduces our considerations to rotation vectors of periodic orbits.

Suppose $\xi \in X$ is a periodic point of period n. We may assume that $\xi = (\xi_1, ..., \xi_n, ...)$ and $(\xi_1, ..., \xi_n)$ is maximally partitioned into k blocks of sizes $n_1,...,n_k$ such that $n_1+...+n_k=n$, and each block exclusively contains elements of either S_1 or S_2. It follows from the construction of Φ that $n\cdot \mathrm{rv}_\Phi(\xi)$ (where $\mathrm{rv}_\Phi(\xi)$ denotes the rotation vector of the unique invariant measure supported on the orbit of ξ) is the sum of blocks of vectors of the form

$$(\alpha-1)w_0+\sum_{i=1}^{n_j-(\alpha-1)} v_s(i). \tag{23}$$

Here $s=1$ if the elements of j^{th} block are from S_1 and $s=2$ if the elements of j^{th} block are from S_2. In case $n_j\le\alpha-1$ the block's contribution is n_jw_0.

First we show that $\mathrm{rv}_\Phi(\xi)\in\overline{\mathrm{Conv}}\{w_s(j): j\ge\alpha,\ s=1,2\}$ for $k=2$ and $n_1,n_2\ge\alpha$. In this case we have

$$\mathrm{rv}_\Phi(\xi)=\frac{1}{n}\begin{pmatrix}(\alpha-1)(x_1+x_2)+\sum\limits_{i=1}^{n_1-\alpha+1}x_i+\sum\limits_{i=1}^{n_2-\alpha+1}x_i\\ \sum\limits_{i=1}^{n_1-\alpha+1}l(x_i)-\sum\limits_{i=1}^{n_2-\alpha+1}l(x_i)\end{pmatrix}. \tag{24}$$

By symmetry we restrict ourselves to the case when $\mathrm{rv}_\Phi(\xi)$ is above or on the x-axis, that is $n_1\ge n_2$. We compare $\mathrm{rv}_\Phi(\xi)$ with points $w_1(n)$ and $\frac{n_1}{n}w_1(n_1)+\frac{n_2}{n}w_1(n_2)$. In the case when $n_2>\alpha$ we have

$$w_1(n)=\frac{1}{n}\begin{pmatrix}\frac{\alpha}{2}(x_1+x_2)+\sum\limits_{i=1}^{n-\alpha}x_i\\ \sum\limits_{i=1}^{n-\alpha}l(x_i)\end{pmatrix}, \tag{25}$$

$$\frac{n_1}{n}w_1(n_1)+\frac{n_2}{n}w_1(n_2)=\frac{1}{n}\begin{pmatrix}\alpha(x_1+x_2)+\sum\limits_{i=1}^{n_1-\alpha}x_i+\sum\limits_{i=1}^{n_2-\alpha}x_i\\ \sum\limits_{i=1}^{n_1-\alpha}l(x_i)+\sum\limits_{i=1}^{n_2-\alpha}l(x_i)\end{pmatrix} \tag{26}$$

Clearly, $\mathrm{pr}_y\left(\frac{n_1}{n}w_1(n_1)+\frac{n_2}{n}w_1(n_2)\right)\ge \mathrm{pr}_y(w_n)>\mathrm{pr}_y(\mathrm{rv}_\Phi(\xi))$. To compare the x-coordinates we use the fact that x_i are decreasing and see that

$$\mathrm{pr}_x(w_n)\le\frac{\alpha}{2}(x_1+x_2)+\sum_{i=1}^{n-2\alpha+2}x_i+(\alpha-2)x_3\le \mathrm{pr}_x(\mathrm{rv}_\Phi(\xi))$$

and

$$\begin{aligned}\mathrm{pr}_x(\mathrm{rv}_\Phi(\xi))&=\mathrm{pr}_x\left(\frac{n_1}{n}w_1(n_1)+\frac{n_2}{n}w_1(n_2)\right)+x_{n_1-\alpha+1}+x_{n_2-\alpha+1}-(x_1+x_2)\\&\le \mathrm{pr}_x\left(\frac{n_1}{n}w_1(n_1)+\frac{n_2}{n}w_1(n_2)\right).\end{aligned}$$

In the case $n_1>n_2=\alpha$ the point $\frac{n_1}{n}w_1(n_1)+\frac{\alpha}{n}w_1(\alpha)$ is still to the right and above $\mathrm{rv}_\Phi(\xi)$ whereas point $w_1(n)$ is to the left and above it. Finally, when $n_1=n_2=\alpha$ we obtain $\mathrm{rv}_\Phi(\xi)=\frac{(\alpha-1)w_0+(x_1,0)}{\alpha}$, which is the mid-point between $w_1(\alpha)$ and $w_2(\alpha)$. It follows that, $\mathrm{rv}_\Phi(\xi)\in\overline{\mathrm{Conv}}\{w_i(j): j\ge\alpha,\ i=1,2\}$.

The case $k = 3$ is similar. We have $n = n_1 + n_2 + n_3$ with $n_1, n_2, n_3 \geq \alpha$. By symmetry, we may assume that $\mathrm{rv}_\Phi(\xi)$ is above or on the x-axis and that we can write

$$\mathrm{rv}_\Phi(\xi) = \frac{1}{n}\left[3(\alpha-1)w_0 + \sum_{i=1}^{n_1-\alpha+1} v_1(i) + \sum_{i=1}^{n_2-\alpha+1} v_2(i) + \sum_{i=1}^{n_3-\alpha+1} v_1(i)\right], \tag{27}$$

where $n_1 \geq n_3$. We compare $\mathrm{rv}_\Phi(\xi)$ with points $\frac{n_1+n_2}{n}w_1(n_1+n_2)+\frac{n_3}{n}w_1(n_3)$ and $\frac{n_1}{n}w_1(n_1) + \frac{n_2}{n}w_1(n_2) + \frac{n_3}{n}w_1(n_3)$. Since $l(x_i)$ is a decreasing sequence, we have

$$\mathrm{pr}_y(\mathrm{rv}_\Phi(\xi)) \leq \mathrm{pr}_y\left(\frac{n_1+n_2}{n}w_1(n_1+n_2) + \frac{n_3}{n}w_1(n_3)\right) \tag{28}$$

$$\leq \mathrm{pr}_y\left(\frac{n_1}{n}w_1(n_1) + \frac{n_2}{n}w_1(n_2) + \frac{n_3}{n}w_1(n_3)\right). \tag{29}$$

To compare the x-coordinates we first consider the case when all n_j are strictly greater than α. Then

$$\mathrm{pr}_x(\mathrm{rv}_\Phi(\xi)) - \mathrm{pr}_x\left(\frac{n_1+n_2}{n}w_1(n_1+n_2) + \frac{n_3}{n}w_1(n_3)\right) \tag{30}$$

$$= (\alpha-3)\frac{x_1+x_2}{2} - \sum_{i=n_1-\alpha+2}^{n_1-\alpha+2} x_i + \sum_{i=1}^{n_2-\alpha+1} x_i + x_{n_3-\alpha+1} \tag{31}$$

$$\geq (\alpha-3)\frac{x_1+x_2}{2} - (\alpha-2)x_{n_1-\alpha+2} + x_{n_3-\alpha+1} \tag{32}$$

Since $n_1 \geq n_3$ and the sequence (x_i) is strictly decreasing we see that $x_{n_1-\alpha+2} < x_{n_3-\alpha+1}$ as well as $x_{n_1-\alpha+1} < \frac{x_1+x_2}{2}$. Therefore, the expression (32) is positive, and the point $\frac{n_1+n_2}{n}w_1(n_1+n_2) + \frac{n_3}{n}w_1(n_3)$ is to the left of $\mathrm{rv}_\Phi(\xi)$. On the other hand,

$$\mathrm{pr}_x(\mathrm{rv}_\Phi(\xi)) - \mathrm{pr}_x\left(\frac{n_1}{n}w_1(n_1) + \frac{n_2}{n}w_1(n_2) + \frac{n_3}{n}w_1(n_3)\right) \tag{33}$$

$$= x_{n_1-\alpha+1} + x_{n_2-\alpha+1} + x_{n_3-\alpha+1} - 3\cdot\frac{x_1+x_2}{2}, \tag{34}$$

which is negative. Therefore, the point $\frac{n_1}{n}w_1(n_1) + \frac{n_2}{n}w_1(n_2) + \frac{n_3}{n}w_1(n_3)$ is to the right of $\mathrm{rv}_\Phi(\xi)$. The case when some of the n_j are equal to α requires separate consideration since the formula for $w_1(\alpha)$ is different. However, the estimates can be done in a similar way and we omit them here. We obtain

$$\mathrm{pr}_x\left(\frac{n_1+n_2}{n}w_1(n_1+n_2) + \frac{n_3}{n}w_1(n_3)\right) \leq \mathrm{pr}_x(\mathrm{rv}_\Phi(\xi)) \tag{35}$$

and

$$\mathrm{pr}_x(\mathrm{rv}_\Phi(\xi)) \leq \mathrm{pr}_x\left(\frac{n_1}{n}w_1(n_1) + \frac{n_2}{n}w_1(n_2) + \frac{n_3}{n}w_1(n_3)\right) \tag{36}$$

whenever $n_j > \alpha$ for at least for one j. In the case $n_1 = n_2 = n_3 = \alpha$ we have $\mathrm{rv}_\Phi(\xi) = w_1(\alpha)$.

It follows that $\mathrm{rv}_\Phi(\xi) \in \overline{\mathrm{Conv}}\{w_i(j) : j \geq \alpha,\ i = 1, 2\}$.

To conclude the proof we notice that the rotation vector of any periodic orbit can be written as a convex combination of vectors described in the previous two cases and w_0. □

Figure 1 illustrates the rotation set of the potential Φ (see (17)) where the set C is a ellipse $(x-1)^2 + \frac{y^2}{2^2} = 1$, $x_1 = 1$, $x_k = \frac{1}{6^k}$ for $k > 1$ and $\alpha = 3$. Below we plot 1000 data points of this rotation set. The shape of the resulting graph gives it the name 'fish'.

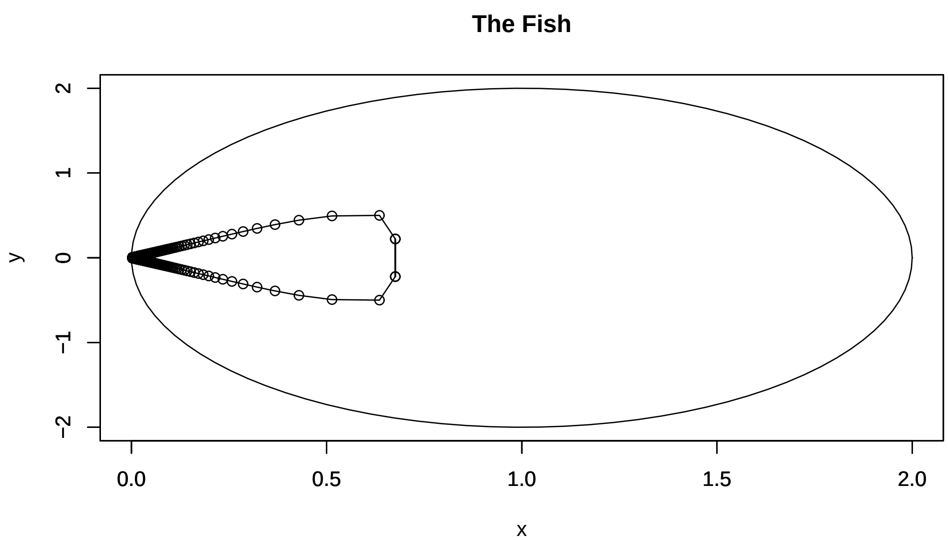

FIGURE 1. The rotation set of the fish based on 1000 data points.

We now list several properties of the system in Example 1 that hold without the symmetry assumptions in Proposition 1.

Theorem 1. *Let X, f and Φ be as in Example 1. Then*

(i) *Φ is Lipschitz continuous;*
(ii) *$\text{int Rot}(\Phi) \neq \varnothing$ and $\text{Rot}(\Phi) \subset \text{int } C \cup \{w_\infty\}$;*
(iii) *There exists precisely two ergodic localized measures of maximal entropy at w_∞;*
(iv) *The entropy function $w \mapsto h_{\text{top}}(w, \Phi, f)$ is real analytic in $\text{int Rot}(\Phi)$.*

Proof. (i) We will work with the $d_{1/2}$ metric (see (16)) on X to show that Φ is Lipschitz continuous. Set $\gamma = \text{diam}(C)$. Let $\xi, \eta \in X$ with $\Phi(\xi) \neq \Phi(\eta)$. If $\Phi(\xi) = w_0$ then $\xi_k \neq \eta_k$ for some $k \leq \alpha$. Hence,

$$\|\Phi(\xi) - \Phi(\eta)\|_2 \leq \gamma = \gamma 2^\alpha \frac{1}{2^\alpha} \leq \gamma 2^\alpha d(\xi, \eta). \tag{37}$$

The case $\Phi(\eta) = w_0$ is analogous. The case $\Phi(\xi) \in l_i \setminus \{w_\infty\}$ and $\Phi(\eta) \in l_j \setminus \{w_\infty\}$ with $i \neq j$ can be treated in a similar way as in (37). It remains to consider the case $\Phi(\xi), \Phi(\eta) \in l_i$ for some $i = 1, 2$. Without loss of generality we assume that $\Phi(\eta)$ is further along on the path to w_∞ as $\Phi(\xi)$. Thus, $d(\xi,\eta) \geq \frac{1}{2^{k_\xi+1}}$ where k_ξ is defined by $\Phi(\xi) = v_i(k_\xi)$. Since $\|v_i(k) - w_\infty\|_2 < 1/2^k$ for all $k \in \mathbb{N}$, we conclude that

$$\|\Phi(\xi) - \Phi(\eta)\|_2 \leq \frac{2}{2^{k_\xi}} \leq 4d(\xi,\eta) \tag{38}$$

which completes the proof of (i).
(ii) Note that points w_0, w_∞ and $\frac{v_1(1)+\alpha w_0}{\alpha+1}$ belong to $\mathrm{Rot}(\Phi)$ and thus $\mathrm{int}\,\mathrm{Rot}(\Phi) \neq \varnothing$.

To prove that $\mathrm{Rot}(\Phi) \subset \mathrm{int}\, C \cup \{w_\infty\}$ we apply again the result of Sigmund that the periodic point measures are weak* dense in $\mathcal{M}$. Suppose $\xi \in X$ is a periodic point of period n and that we have the decomposition (23). We use the notation

$$w_s^*(j) = \frac{(\alpha-1)w_0 + \sum_{i=1}^{j-(\alpha-1)} v_s(i)}{j}, \quad s = 1, 2;\ j \geq \alpha. \tag{39}$$

Then the rotation vector of any periodic orbit can be written as a convex combination of $w_s^*(j)$ and w_0. The fact that the set C is strictly convex implies that $w_s^*(j) \in \mathrm{int}\ C$ for all j. Also, it is easy to see that the $w_s^*(j)$ converge to w_∞ as $j \to \infty$. Indeed,

$$\begin{aligned}\|w_\infty - w_s^*(j)\| &\leq \frac{1}{j}\left((\alpha-1)\|w_0 - w_\infty\| + \sum_{i=1}^{j-\alpha+1} \|v_s(i) - w_\infty\|\right) \\ &\leq \frac{1}{j}\left((\alpha-1)\|w_0 - w_\infty\| + \sum_{i=1}^{j-\alpha+1} \frac{1}{2^i}\right) \\ &\leq \frac{(\alpha-1)\|w_0 - w_\infty\| + 1}{j}.\end{aligned} \tag{40}$$

Since $\{w_i^*(j)\}_{j\geq\alpha} \subset \mathrm{int}\, C$ and w_∞ is their only accumulation point, we have $\overline{\mathrm{Conv}\{w_i^*(j)\}_{j\geq\alpha}} \subset \mathrm{int}\, C \cup \{w_\infty\}$ and thus $\mathrm{Rot}(\Phi) \subset \mathrm{int}\, C \cup \{w_\infty\}$.
(iii) We will compute the logarithmic rate of growth of periodic orbits with rotation vectors in the neighborhood of w_∞. Fix $0 < r < \frac{1}{2}d(w_0, w_\infty)$. Suppose $\xi \in X$ is a periodic point of period n and $\mathrm{rv}_\Phi(\xi) \in D(w_\infty, r)$. We may assume decomposition (23). Since there are k blocks and each block contributes at least one w_0 to $\mathrm{rv}_\Phi(\xi)$ we have

$$\frac{kd(w_0, w_\infty)}{n} < d(\mathrm{rv}_\Phi(\xi), w_\infty) < r. \tag{41}$$

Therefore, $k < \frac{nr}{d(w_0,w_\infty)}$. Denote $m = \lfloor \frac{nr}{d(w_0,w_\infty)} \rfloor$, the largest integer smaller than $\frac{nr}{d(w_0,w_\infty)}$. Note that $m < \frac{1}{2}n$ since $r < \frac{1}{2}d(w_0, w_\infty)$.

The maximal number of points of period n in $D(r, w_\infty)$ is

$$\sum_{k=1}^{m} \binom{n}{k-1} \prod_{j=1}^{k} 2^{n_j} = 2^n \sum_{k=1}^{m} \binom{n}{k-1}. \tag{42}$$

We will estimate $\sum_{k=1}^{m} \binom{n}{k-1} = \binom{n}{m-1} + \binom{n}{m-2} + \ldots + \binom{n}{0}$. We have

$$\begin{aligned} &\frac{\binom{n}{m-1} + \binom{n}{m-2} + \ldots + \binom{n}{0}}{\binom{n}{m}} \\ &= \frac{m}{n-m+1} + \frac{m(m-1)}{(n-m+1)(n-m+2)} + \ldots \\ &\le \frac{m}{n-m+1} + \left(\frac{m}{n-m+1}\right)^2 + \left(\frac{m}{n-m+1}\right)^3 + \ldots \\ &= \frac{m}{n-2m+1}. \end{aligned} \tag{43}$$

In the last equality we used the sum of geometric progression with common ratio $\frac{m}{n-m+1}$ which is less than one since $m < \frac{1}{2}n$. We obtain

$$\sum_{k=1}^{m} \binom{n}{k-1} \le \binom{n}{m} \frac{m}{n-2m+1}. \tag{44}$$

Using the well known fact that $\frac{n^n}{e^{n-1}} \le n! \le \frac{(n+1)^{n+1}}{e^n}$ we obtain

$$\log \binom{n}{m} \le (n+1)\log(n+1) - m\log m - (n-m)\log(n-m). \tag{45}$$

To simplify the notation in the following computation we denote $\rho = \frac{r}{d(w_0, w_\infty)}$. Using $n\rho - 1 < m \le n\rho$, we estimate the growth rate of the periodic orbits of period n in $D(r, w_\infty)$.

$$\begin{aligned} \frac{1}{n}\log 2^n \sum_{k=1}^{m} \binom{n}{k-1} &\le \log 2 + \frac{1}{n}\log\frac{n\rho}{n-3n\rho+3} + \frac{n+1}{n}\log(n+1) \\ &\quad - \frac{n\rho-1}{n}\log(n\rho-1) - \frac{n-n\rho}{n}\log(n-n\rho). \end{aligned} \tag{46}$$

Passing to the limit as n approaches infinity we obtain the growth rate of the periodic orbits

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n}\log 2^n \sum_{k=1}^{m} \binom{n}{k-1} \\ &\le \log 2 + \lim_{n\to\infty} \left[\log(n+1) - \rho\log(n\rho-1) - (1-\rho)\log(n-n\rho)\right] \\ &= \log 2 + \log\rho - (1-\rho)\log(1-\rho). \end{aligned} \tag{47}$$

Note that the last expression is greater than $\log 2$ since $\log(1-\rho) < 0$. Since $\rho \to 0$ as $r \to 0$, we have

$$\lim_{r\to 0}\left[\log 2 + \log\rho - (1-\rho)\log(1-\rho)\right] = \log 2. \tag{48}$$

Therefore, we have $h_{\mathrm{top}}(w_\infty, \Phi, f) = h_{\mathrm{top}}(f_{\mathrm{A}}) = \log 2$. Thus, the two distinct ergodic localized measures of maximal entropy at w_∞ are the two ergodic measures of maximal entopy of f_{A}.
Finally, (vi) is proven [20]. For completeness, we give here a brief outline of the proof. Consider a family of potentials $\varphi_{(t,s)} = t\phi_1 + s\phi_2$, where $(t,s)\in\mathbb{R}^2$ and ϕ_1,ϕ_2 are the coordinate functions of the potential Φ. Denote by $\mu_{(t,s)}$ the unique equilibrium state of the potential $\varphi_{(t,s)}$. Suppose for a point $w\in \mathrm{Rot}(\Phi)$ there exist a vector $(t,s)\in\mathbb{R}^2$ such that $\mathrm{rv}_\Phi(\mu_{(t,s)}) = w$. Note that in such case we have

$$\int \varphi_{(t,s)}\, d\mu_{(t,s)} = \int (t\phi_1 + s\phi_2)\, d\mu_{(t,s)} = (t,s)\cdot \mathrm{rv}_\Phi(\mu_{(t,s)}) = (t,s)\cdot w.$$

Here $(t,s)\cdot w$ denotes the scalar product of the vectors (t,s) and $w = (w_1,w_2)$, i.e. $(t,s)\cdot w = tw_1 + sw_2$. The fact that $\mu_{(t,s)}$ is the equilibrium state of $\varphi_{(t,s)}$ implies that

$$\begin{aligned} P_{\mathrm{top}}(\varphi_{(t,s)}) &= h_{\mu_{(t,s)}}(f) + (t,s)\cdot w \\ &= \sup_{\mu\in\mathcal{M}}\{h_\mu(f) + (t,s)\cdot \mathrm{rv}_\Phi(\mu)\} \\ &= \sup_{\mu\in\mathcal{M}_\Phi(w)}\{h_\mu(f) + (t,s)\cdot w\} \end{aligned}$$

and hence $h_{\mu_{(t,s)}}(f) = \sup\{h_\mu(f) : \mu\in\mathcal{M}_\Phi(w)\} = h_{\mathrm{top}}(w)$. Since the potential Φ is Lipschitz continuous, the pressure function $(t,s)\mapsto P_{\mathrm{top}}(\varphi_{(t,s)})$ is real-analytic on $\mathbb{R}^2$ and its gradient vector is precisely the rotation vector of $\mu_{(t,s)}$ (see [15, Lemma 1]). It follows that the functions $R(t,s) = \mathrm{rv}_\Phi(\mu_{(t,s)})$ and $H(t,s) = h_{\mu_{(t,s)}}(f)$ are also real-anaclitic. By using methods from the thermodynamic formalism we are now able show that $R:\mathbb{R}^2\to \mathrm{int}\,\mathrm{Rot}(\Phi)$ is a C^ω-diffeomorphism (see [20] for a detailed proof). We conclude that

$$w\mapsto h_{\mathrm{top}}(w) = H(R^{-1}(w))$$

is real-analytic in the interior of $\mathrm{Rot}(\Phi)$. □

Remarks. (i) *Note that our techniques can be easily adapted to construct for any given $n\in\mathbb{N}$ examples with precisely n ergodic localized measures of maximal entropy at a boundary point of the rotation set.*
(ii) *Note that in Example 1, we have two ergodic entropy maximizing measures at w_∞ which is an exposed point of* $\mathrm{Rot}(\Phi)$. *Recall that a point w is an exposed point of a convex subset $K\in\mathbb{R}^m$ if there is a supporting hyperplane for K through w which does not contain any other point of K. The line through w_∞ orthogonal to the segment $[w_\infty, w_0]$ is supporting for the set C, and hence for* $\mathrm{Rot}(\Phi)$. *Moreover, w_∞ is the only point of* $\mathrm{Rot}(\Phi)$ *in this line, since the boundary of C is strictly convex.*

(iii) *The cardinality of ergodic localized equilibrium states at the boundary is in general not preserved under small changes of the potential* Φ*. Indeed, for any* $\varepsilon > 0$ *pick a point* $w_\varepsilon \in \partial C$ *such that* $w_\varepsilon \neq w_\infty$ *and* $dist(w_\infty, w_\varepsilon) < \varepsilon$*. Redefine* $v_2(k)$ *within an* ε*-neighbourhood so that* $|w_\varepsilon - v_2(k)| < \frac{1}{2^k}$*. Define*

$$\Phi_\varepsilon(\xi) = \begin{cases} w_0 & \text{if } \xi \in Y_0(\alpha) \\ v_i(k-\alpha) & \text{if } \xi \in Y_i(k-1) \text{ and } x \notin Y_i(k),\ k > \alpha \\ w_\infty & \text{if } \ \xi \in Y_1(k) \text{ for all } k \\ w_\varepsilon & \text{if } \ \xi \in Y_2(k) \text{ for all } k \end{cases} \tag{49}$$

Clearly $\|\Phi - \Phi_\varepsilon\| < \varepsilon$*. However,* Φ_ε *has a unique (ergodic) localized measure of maximal entropy at* w_∞ *and* Φ *has precisely two ergodic localized measures of maximal entropy at* w_∞*.*

References

1. L. Alseda, J. Llibre, F. Maiiosas and M. Misiurewicz *Lower bounds of the topological entropy for continuous maps of the circle of degree one*, Nonlinearity **1** (1988) 463–479.
2. L. Alseda and F. Manosas *Kneading theory and rotation intervals for a class of circle maps of degree one*, Nonlinearity **3** (1990) 413–452.
3. A. Blokh, *Functional rotation numbers for one dimensional maps*, Trans. Amer. Math. Soc., **347** (1995), 499-513.
4. L. Block, J. Guckenheimer, M. Misiurewicz and L. S. Young *Periodic points and topological entropy of one dimensional maps*, Global Theory of Dynamical Systems Lecture Notes in Mathematics **819** (1980), 18–34.
5. T. Bousch, *Le poisson n'a pas d'aretes*, Annales de l'Institut Henri Poincaré (probabilités et statistiques) **36** (2000), 489–508.
6. R. Bowen, *Some systems with unique equilibrium states*, Math. Systems Theory **8** (1974/75), 193-202.
7. R. Bowen *Entropy expansive maps*, Trans. Am. Math. Soc. **164** (1972), 323–331.
8. P. Boyland, A. Carvalho and T. Hall, *New rotation sets in a family of torus homeomorphisms*, preprint.
9. N. Chernov, *Invariant measures for hyperbolic dynamical systems*, Handbook of Dynamical Systems, Vol 1A, B. Hasselblatt and A. Katok, eds, Elsevier, Amsterdam (2002), 321–447.
10. T. Fisher, L. Diaz, M. Pacifico and J. Vieitez, *Entropy-expansiveness for partially hyperbolic diffeomorphisms*, Discrete Contin. Dyn. Syst. **32** (2012), 4195-4207.
11. J. Franks *Realizing rotation vectors for torus homeomorphisms*, Trans. Amer. Math. Soc. **311** (1989), 107–115.
12. W. Geller and M. Misiurewicz, *Rotation and entropy*, Trans. Amer. Math. Soc., **351** (1999), 2927-2948.
13. N. Haydn and D. Ruelle, *Equivalence of Gibbs and equilibrium states for homeomorphisms satisfying expansiveness and specification*, Comm. Math. Phys. **148** (1992), 155-167.
14. M. R. Herman, *Existence et non existence de tores invariants par des diffeomorphismes symplectiques*, Sminaire quations aux drives partielles (dit "Goulaouic-Schwartz") **(1987-1988)**, 1–24.
15. O. Jenkinson, *Rotation, entropy, and equilibrium states*, Trans. Amer. Math. Soc. **353** (2001), 3713–3739.

16. O. Jenkinson, *Geometric Barycentres of Invariant Measures for Circle Maps*, Ergodic Theory and Dynamical Systems **21** (2001), 511–532.
17. A. Katok, *Bernoulli diffeomorphisms on surfaces*, Ann. of Math. (2) **110** (3) (1979), 529–547.
18. A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems. With a supplementary chapter by Katok and Leonardo Mendoza*, Encyclopedia of Mathematics and its Applications, **54**, Cambridge University Press, Cambridge, 1995.
19. T. Kucherenko, *Localized Variational Principle for Non-Besicovitch Metric Spaces*, Topology and its Applications, **190**, (2015), 22–30.
20. T. Kucherenko and C. Wolf, *Geometry and entropy of generalized rotation sets*, Israel Journal of Mathematics, **199** (2014), 791–829.
21. T. Kucherenko and C. Wolf, *Localized Pressure and equilibrium states*, Journal of Statistical Phisics, to appear.
22. J. Kwapisz, *Every convex polygon with rational vertices is a rotation set*, Ergodic Theory and Dynamical Systems **12** (1992), 333–339.
23. J. Kwapisz, *A toral diffeomorphism with a nonpolygonal ratation set*, Nonlinearity 8, (1995), 461-476.
24. J. Kwapisz, *A priori degeneracy of one-dimensional rotation sets for periodic point free torus maps*, Trans. Amer. Math. Soc. 354 (2002), 2865-2895
25. M. Misiurewicz and K. Ziemian, *Rotation Sets for Maps of Tori*, J. London Math. Soc. **s2-40 (3)** (1989), 490–506.
26. M. Misiurewicz and K. Ziemian, *Rotation sets and ergodic measures for torus homeomorphisms*, Fundam. Math. **137** (1991), 45–52.
27. S. Newhouse, *Continuity properties of entropy*, Ann. of Math. (2) **129** (1989), 215–235.
28. A. Passeggi, *Rational polygons as rotation sets of generic homeomorphisms of the two torus*, J. London Math. Soc. (1) **89** (2014), 235-254
29. H. Poincaré, *Sur les cousbes définies par les équations différentielles*, Euvres Complètes, tome 1, Gauthier-Villars, Paris, (1952), 137-158.
30. D. Ruelle, *Thermodynamic Formalism*, Cambridge: Cambridge Univ. Press, 2004.
31. P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics 79, Springer, 1981.
32. K. Ziemian, *Rotation sets for subshifts of finite type*, Fund. Math. **146** (1995), 189–201.

Department of Mathematics, The City College of New York, New York, NY, 10031, USA
E-mail address: tkucherenko@ccny.cuny.edu

Department of Mathematics, The City College of New York, New York, NY, 10031, USA
E-mail address: cwolf@ccny.cuny.edu