V. Kadets, T. Kucherenko, Weak topology and properties fulfilled almost everywhere, *Mat. Fiz. Anal. Geom.*, 2001, Volume 8, Number 3, 261–271

Matematicheskaya fizika, analiz, geometriya
2001, v. 8, No. 3, p. 261–271

Weak topology and properties fulfilled almost everywhere

V. Kadets and T. Kucherenko

Department of Mechanics and Mathematics, V.N. Karazin Kharkov National University
4 Svobody Sq., Kharkov, 61077, Ukraine
E-mail:vishnyakova@ilt.kharkov.ua
kucherenko_toma@yahoo.com

Received March 20, 2001
Communicated by G.M. Feldman

Let B be a Banach space. A sequence of B-valued functions $\langle f_n \rangle$ is weakly almost everywhere convergent to 0 provided $x^* \circ f_n$ is almost everywhere convergent to 0 for every continuous linear x^* on B. A Banach space is finite dimensional if and only if every weakly almost everywhere convergent sequence of B-valued functions is almost everywhere bounded. If B is separable, B^* is separable if and only if every weakly almost everywhere convergent to 0 and almost everywhere bounded sequence of B-valued functions is weakly convergent to 0 almost everywhere.

1. Introduction

There are two natural ways to introduce 'properties fulfilled almost everywhere' in weak topology of Banach space. One way is to apply a linear functional to a vector-valued function and then consider the scalar property fulfilled almost everywhere. The other way is to delete at first an exclusive set and then consider the weak property in points of the remained set. Thus we can give two definitions.

Definition 1.1. *Let B be a Banach space. A function $f : [0,1] \to B$ is said to be weakly almost everywhere equal 0 provided that, for any x^* in the continuous dual B^* of B, the function $x^* \circ f$ equals 0 almost everywhere.*

Definition 1.2. *A function $f : [0,1] \to B$ is said to be weakly almost everywhere equal* 0 *in alternative sense if there exists a set A of full measure, such that for any $x^* \in B^*$ function $x^* \circ f(t)$ equals 0 at any point $t \in A$.*

Mathematics Subject Classififcations 2000: 46B15, 46A35, 54A20.
Key words: weak convergence; weak almost everywhere convergence; separable duals.
The research was conducted during the stay in the Free University of Berlin under the support of the Alexander von Humboldt Foundation.

It is obvious that if a function f satisfies the Definition 1.2, then f equals 0 weakly almost everywhere by Definition 1.1. Further we give an example, that the converse statement is not true in general.

Definition 1.2 is equivalent to that f equals 0 almost everywhere. Thus below when we speak about weakly almost everywhere 0 functions, we shall keep in mind the Definition 1.1.

In the first part of this paper we study the connection between the Definitions 1.1 and 1.2. We consider a wide class of Banach spaces in which these Definitions are equivalent and we present examples of Banach spaces in which it is not true. In particular, it is shown that if for any continuum subspace of a Banach space B there exists a total sequence of functionals, then Definition 1.1 and Definition 1.2 are equivalent for every B-valued function. Now suppose there is a continuum subspace of B without any total sequence, that has a continuum dual. Then under the assumption of the continuum hypothesis the Definition 1.1 does not yield Definition 1.2. And finally, we study conditions under which the Definitions 1.1 and 1.2 are equivalent for Borel measurable functions.

In the second part of the paper we consider the notion of weak almost everywhere convergence for the sequences of functions. The aim of this section is to give a characterization of finite dimensional spaces and separable Banach spaces with separable duals via weak almost everywhere convergence.

We close this section by collecting some notations. Throughout the paper, B, D, X, Y, and Z denote Banach spaces or their subspaces. We cosider finite as well as infnite-dimensional spaces, but the trivial case of 0-dimensional space is excluded from our consideration. $\mathbb{N}$ and $\mathbb{R}$ are the sets of all naturals and all reals respectively. For a given set $\Gamma, l_1(\Gamma)$(respectively $l_2(\Gamma)$) is the space of all scalar valued functions f on Γ with $\sum_{t\in\Gamma} |f(t)| < \infty$ (respectively $\sum_{t\in\Gamma} |f(t)|^2 < \infty$). $\sum_{t\in\Gamma} |f(t)|$ is the norm in $l_1(\Gamma)$ and $\left(\sum_{t\in\Gamma} |f(t)|^2\right)^{\frac{1}{2}}$ is the norm in $l_2(\Gamma)$. If $\Gamma = \mathbb{N}$ we use the notations l_1 and l_2. The space of all scalar sequences we denote by $\mathbb{R}^\omega$. All the Banach space-valued functions below are defined on the unit interval $[0,1]$ equiped with the Lebesque measure μ.

2. Weak equality almost everywhere

Let B be a Banach space. A B-valued function f is said to be weakly almost everywhere equal 0 provided that, for any x^* in the continuous dual B^* of B, the function $x^* \circ f$ equals 0 almost everywhere.

To put it in another way for any $x^* \in B^*$ there is a negligible set $A_{x^*} \subset [0,1]$ such that $x^* \circ f(t) = 0$ for every point $t \in [0,1] \setminus A_{x^*}$.

The sets A_{x^*} generally are different for different functionals x^*. If it is possible to select a common for all functionals negligible set, outside which the function

f will be vanished on each linear functional, then by Hahn–Banach theorem the function f equals 0 outside this set. So $f = 0$ almost everywhere.

Generally it doesn't follow from function's equality to 0 weakly, a.e., that f equals 0 almost everywhere.

Example 2.1. *A weakly almost everywhere zero function, which doesn't equal 0 at any point.*

Consider the space $l_2([0,1])$. For every $t \in [0,1]$ denote by e_t the characteristic function of point t, $e_t(t) = 1$, and $e_t(x) = 0$ for all the other points x. Since for any $x \in l_2([0,1])$ $x = \sum\limits_{t \in [0,1]} x(t)\, e_t$, $\{e_t\}_{t \in [0,1]}$ is the uncountable basis in $l_2([0,1])$. Define the function $f : [0,1] \to l_2([0,1])$ by the rule $f(t) = e_t$. Take an arbitrary $x^* \in l_2^*([0,1])$. Then $\langle x^*, f(t)\rangle = \langle x, e_t\rangle = x(t)$. Since the support of $x(t)$ is at most countable and the measure of any countable set equals 0, it follows that $\langle x^*, f\rangle \overset{\text{a.e.}}{=} 0$. Therefore $f = 0$ weakly almost everywhere. But $\|f(t)\| = \|e_t\| = 1$, so $f(t) \neq 0$ for any $t \in [0,1]$.

Definition 2.1. *Let B be a Banach space. A sequence of functionals $\{x_n^*\}_{n=1}^{\infty}$ is said to be total if for any non-zero $x \in B$ there exist $n \in \mathbb{N}$ such that $x_n^*(x) \neq 0$.*

It is easy to check, that the dual of every separable space contains a total sequence. There are some more nonseparable spaces, whose duals contain a total sequence. In particular the space l_∞ isn't separable but the coordinate functionals in l_∞ form a total sequence. For a reflexive space B the existence of a countable total subset of B^* is equivalent to separability of B.

Proposition 2.2. *Let B be a Banach space and $f : [0,1] \to B$. Suppose B^* contains a total sequence and $f = 0$ weakly almost everywhere. Then f is almost everywhere 0.*

P r o o f. Let $\{x_n^*\}_{n=1}^{\infty}$ be a total sequence in B^*. Then for any $n \in \mathbb{N}$ there exists a subset $A_n \subset [0,1]$ such that $\mu(A_n) = 0$ and $x_n^* \circ f = 0$ outside A_n. Now consider the set $A = \bigcup\limits_{n=1}^{\infty} A_n$. Note that A is also of zero measure. To conclude the proof it remains to show that $f = 0$ outside A. Take an arbitrary point $t \notin A$. Suppose $f(t) \neq 0$. Then there exists $n \in \mathbb{N}$ such that $\langle x_n^*, f(t)\rangle \neq 0$, contrary to the construction of the set A. ∎

If the dual of a Banach space contains a total sequence, then the cardinality of B can't be too large.

Remark 2.1. *Suppose B is a Banach space and its dual contains a total sequence. Then the cardinality of B equals continuum.*

P r o o f. Let $\{x_n^*\}_{n=1}^{\infty}$ be the total sequence of functionals in B^*. By x denote an arbitrary element of B . Clearly if $\langle x_n^*, x\rangle = 0$ for any $n \in \mathbb{N}$ then $x = 0$. Define a linear map $F : B \to \mathbb{R}^{\omega}$ by the rule

$$F(x) = \{x_1^*(x), x_2^*(x), ..., x_n^*(x), ...\}.$$

Let us show that F is injective. Suppose $F(x) = 0$. Then $x_n^*(x) = 0$ for any $n \in \mathbb{N}$, so $x = 0$.

This implies that the cardinality of B doesn't exceed the cardinality of $\mathbb{R}^{\omega}$. Since the cardinality of $\mathbb{R}^{\omega}$ equals continuum, the cardinality of B is also continuum. ■

Remark 2.2. *Consider an arbitrary Banach space B and a function $f : [0,1] \to B$. Let $f([0,1])$ be the set of all values of the function f. By X denote the closure of the linear span of $f([0,1])$. Cardinality of X is at most continuum. So we can restrict our considerations only for spaces of continuum cardinality.*

The rest of this section is devoted to converse in some sense statements to Proposition 2.2. These converse statements need additional assumptions on cardinality of B^*. Moreover this statements depend on continuum hypothesis or Martin's axiom.

Before we consider the converse case of this statement, it will be useful to introduce some notation. Denote by K_0 the second numeral class. This means that K_0 is the set of all ordinal numbers, being types of countable well-ordered sets. Let us remind that continuum hypothesis states, that the cardinality of K_0 is continuum. Let ω be the least number in K_0 (ω is the type of $\mathbb{N}$), and let Ω be the first number following K_0. The ordinal number is of the first kind (respectively of the second kind) if it has (respectively if it doesn't have) the direct predecessor.

Lemma 2.3. *Suppose A is an arbitrary continuum set and $\{A_\alpha\}_{\alpha\in\mathfrak{A}}$ is a continuum family of its subsets, such that for any countable sequence of indices $\{\alpha_k\}_{k=1}^{\infty} \subset \mathfrak{A}$ the cardinality of $\bigcap\limits_{k=1}^{\infty} A_{\alpha_k}$ is equal to continuum. Then under the assumption of continuum hypothesis there exists a set $B \subset A$ such that B has cardinality of continuum and the set $B \setminus A_\alpha$ is no more than countable for all $\alpha \in A$.*

P r o o f. We may assume that our family of sets is numbered by ordinal numbers $\alpha < \Omega$. We shall construct a sequence of at most countable sets $F(\alpha) \subset A$, $\alpha < \Omega$. The required set B will be $\bigcup_{\alpha<\Omega} F(\alpha)$.

Let a be an arbitrary point of A_0, set $F(0) = \{a\}$. Suppose $\alpha < \omega$ and we constructed $F(\alpha - 1)$. Since the cardinality of $\bigcap_{\beta=0}^{\alpha} A_\beta$ is continuum, there exist $a \in \bigcap_{\beta=0}^{\alpha} \Big(A_\beta \setminus F(\alpha-1)\Big)$. Set $F(\alpha) = F(\alpha-1) \bigcup \{a\}$. In the same way we construct $F(\alpha)$, $\alpha < \omega$. Now we set $F(\omega) = \bigcup_{\beta<\omega} F(\beta)$. For $\alpha = \omega + 1$ we set $F(\alpha) = F(\omega) \bigcup \{a\}$, where $a \in \bigcap_{\beta\le\alpha} A_\beta \setminus F(\omega)$, and so on. In other words, if α is the number of the first kind, we put $F(\alpha) = F(\alpha - 1) \bigcup \{a\}$ where $a \in \bigcap_{\beta\le\alpha} A_\beta \setminus F_\alpha$. It is possible because the cardinality of $\bigcap_{\beta\le\alpha} A_\beta$ is continuum and $F(\alpha)$ is at most countable. If α is the number of the second kind we put $F(\alpha) = \bigcup_{\beta<\alpha} F(\beta)$. Now we show that $B = \bigcup_{\alpha<\Omega} F(\alpha)$ is the required set. At first, since B is a union of a continuum family of different sets, the cardinality of B is continuum. Now for a fixed $\beta < \Omega$ let a be an arbitrary point of $B \setminus A_\beta$. Since $a \notin A_\beta$, a would be added to B at the step number of which is less than β. Therefore $B \setminus A_\beta$ is at most countable. ■

Theorem 2.4. *Suppose the dual of a Banach space B has the continuum cardinality. Then under the assumption of continuum hypothesis the following conditions are equivalent:*

(i) every weakly almost everywhere 0 B-valued function is strongly almost everywhere 0;

(ii) there exists a total sequence in B^*.

P r o o f. Note that Proposition 2.2 provides $(ii) \mapsto (i)$. In order to proof $(i) \mapsto (ii)$, we assume that there are no total sequences in B^* and construct a weakly almost everywhere 0 function $f : [0,1] \to B$ which is not equal to 0.

Consider a family of sets $\{B_\alpha\}_{\alpha\in B^*}$ in $B \setminus \{0\}$, where $B_\alpha = \ker\alpha \setminus \{0\}$. $\{B_\alpha\}_{\alpha\in B^*}$ is a continuum family of continuum sets. Let $\{\alpha_k\}_{k=1}^{\infty} \subset B^*$ be an arbitrary countable subset. Since $\{\alpha_k\}_{k=1}^{\infty}$ is not total, there exists a non-zero element $x \in B$ such that $\alpha_k(x) = 0$ for all k. Therefore $x \in \bigcap_{k=1}^{\infty} B_{\alpha_k}$. But then any element of the form $a \cdot x$ belongs $\bigcap_{k=1}^{\infty} B_{\alpha_k}$ where $a \in \mathbb{R} \setminus \{0\}$. This implies that the cardinality of $\bigcap_{k=1}^{\infty} B_{\alpha_k}$ is continuum. Now by the Lemma 2.3, select $B_0 \subset B \setminus \{0\}$, such that cardinality of B_0 is continuum and for any $\alpha \in B^* \setminus \{0\}$ the set $B_0 \setminus \ker \alpha$ is at most countable. We may establish the one-to-one correspondence between the set B_0 and the segment [0,1] . Denote by f the one-to-one map of [0,1] onto B_0. For any $\alpha \in B^*$

$$\{t : \langle \alpha, f(t) \rangle \neq 0\} = \{t : f(t) \in B_0 \setminus \ker \alpha\}.$$

Since $B_0 \setminus \ker \alpha$ is at most countable, the set $\{t : \langle \alpha, f(t) \rangle \neq 0\}$ is also at most countable and its measure equals 0. It follows that f is weakly almost everywhere 0. On the other hand the function f doesn't equal 0 at any point. ■

Corollary 2.1. *Let B be a reflexive Banach space. Then under the assumption of continuum hypothesis B is separable iff every weakly almost everywhere 0 B-valued function is strongly almost everywhere 0.*

The proof follows immediately from the previous theorem and the fact that every nonseparable space has a nonseparable subspace of continuum cardinality.

Consider the space $l_1([0,1])$. It is a space of continuum cardinality and the cardinality of its dual is more than continuum. Polynomials with rational coefficients form a total sequence in $(l_1([0,1]))^*$. Thus every almost everywhere 0 function $f : [0,1] \to l_1([0,1])$ equals 0 almost everywhere. This implies that the continuum cardinality of the dual is not a necessary condition in Theorem 2.4. The following lemma shows that we don't need to consider the duals of more than continuum cardinality if we restrict our considerations only to the case of weakly Borel measurable functions.

Lemma 2.5. *Suppose $f : [0,1] \to B$ is weakly Borel measurable. If B is the closure of the linear span of the set $f([0,1])$, then the cardinality of B^* is at most continuum.*

P r o o f. Since the cardinality of the collection of all the Borel subsets of $[0,1]$ is continuum, the space of Borel measurable scalar functions has also the continuum cardinality. Let F be the map of B^* to the space of Borell measurable scalar functions such that $F(x^*) = \langle x^*, f(t) \rangle$ for all $x^* \in B^*$. We shall show that F is an injection. Suppose $F(x_1^*) = F(x_2^*)$, where $x_1^*, x_2^* \in B^*$. Then $\langle x_1^*, f(t) \rangle = \langle x_2^*, f(t) \rangle$ for all $t \in [0,1]$. Therefore the values of x_1^* coincide with the values of x_2^* on the linear span of $f([0,1])$. Since the linear span of $f([0,1])$ is a dense subset of B, the functional x_1^* is equal to x_2^*. It follows that the cardinality of B^* is no more than the cardinality of the space of Borel measurable scalar functions. ■

The assumption of continuum hypothesis in Theorem 2.4 is rather restrictive. Under the assumption of less restrictive axioms (like, for example, Martin's axiom) some other results can be obtained.

Martin's axiom in particular yields that the union of less then continuum collection of sets of reals of Lebesque measure zero is of measure zero itself (see [4]). If the continuum hypothesis is rejected, but the Martin's axiom is assumed, then $l_2(\Gamma)$ for $\Gamma = \aleph_1$ will have no countable total systems, but w-almost everywhere 0 function is 0 a.e. In this case the condition (ii) of the Theorem 2.4. must be substituted by (ii'): B^* has a total set of less than continuum cardinality. This statement can be proved in much the same way as Theorem 2.4.

3. Weak convergence almost everywhere

The results presented in this section have analogues for statistical convergence of sequences (see [1]). Some ideas of [1] are also used below.

Definition 3.1. *Let B be a Banach space, let f_n, $n \in \mathbb{N}$, and f be B-valued functions. The sequence f_n is weakly almost everywhere convergent to f provided that for any x^* in the continuous dual B^* of B, the sequence of scalar-valued functions $x^*(f_n - f)$ is convergent to 0 almost everywhere.*

Proposition 3.2. *Let B be a Banach space with a separable dual B^*. Then every weakly almost everywhere convergent to zero pointwise bounded sequence of B-valued functions f_n defined on [0,1], is weakly convergent to 0 almost everywhere.*

P r o o f. Let D be a countable dense subset of B^* and for each $d^* \in D$ select a subset $A_{d^*} \subset [0,1]$ such that $\mu(A_{d^*}) = 0$ and outside A_{d^*} $\quad d^* \circ f_n \to 0$. Denote

$$A = \bigcup_{d^* \in D} A_{d^*}$$

The set A is also of zero measure and for each $d^* \in D \qquad d^* \circ f_n(t) \to 0$ for all $t \notin A$. The remainder of the proof demonstrates that outside A sequence f_n is weakly convergent to 0 . The condition of boundednes f_n means that $\sup_n \|f_n(t)\| \le C(t)$, where $C(t)$ is a positive finite function.
Fix $t \notin A$, $x^* \in B^*$ and an $\varepsilon > 0$. Select $d^* \in D$ such that $\|d^* - x^*\| < \frac{\varepsilon}{2C(t)}$. There is an $N_\varepsilon \in \mathbb{N}$ such that if $n \ge N_\varepsilon$ then $|d^* \circ f_n(t)| < \frac{\varepsilon}{2}$. The triangle inequality now yields that $|x^* \circ f_n(t)| < \varepsilon$ whenever $n > N_\varepsilon$. That is $x^* \circ f_n(t) \to 0$ and so f_n weakly converge to 0 on the set $[0,1] \setminus A$. ∎

The main results of this section are:

Theorem 3.3. *Let B be a Banach space. Then B is finite dimensional if and only if every weakly almost everywhere convergent sequence of B-valued functions is bounded almost everywhere.*

Theorem 3.4. *Let B be a separable Banach space. Then B has a separable dual if and only if every weakly almost everywhere convergent and pointwise bounded sequence of functions is weakly convergent almost everywhere.*

Proofs of the main results. First note that in a finite-dimensional space all the norms are equivalent and the weak and norm topologies agree. So in a finite dimensional spaces every weakly almost everywhere convergent sequence of functions is norm convergent almost everywhere and so is almost everywhere bounded. Next we show that any infinite dimensional Banach space contains a sequence of functions, which is not bounded at almost all points, but weakly almost everywhere converges to 0.

By Dvoretzky's "almost eucledian sections" theorem in any infinite-dimensional Banach space B for every integer n there exists a collection of elements $\{x_{j,n}\}_{j=1}^{9^n}$ in B such that

$$\left(\sum_{j=1}^{9^n}|a_j|^2\right)^{\frac{1}{2}} \leq \left\|\sum_{j=1}^{9^n}a_jx_{j,n}\right\| \leq 2\left(\sum_{j=1}^{9^n}|a_j|^2\right)^{\frac{1}{2}} \tag{3.1}$$

for all collections of numbers $\{a_j\}_{j=1}^{9^n}$. Observe that by (3.1) for every $h^* \in B^*$

$$\Big(\sum_{j=1}^{9^n}|\langle h^*, x_{j,n}\rangle|^2\Big)^{\frac{1}{2}} \leq 2\|h^*\|. \tag{3.2}$$

Let $r_j(t)$ be the Rademacher functions on [0,1], that is $r_j(t) = sign\sin 2^j\pi t$, where $t \in [0,1]$. Define B-valued functions f_n by

$$f_n(t) = \frac{1}{2^n}\sum_{j=1}^{9^n}r_j(t)x_{j,n}$$

By orthonormality of the Rademacher system and (3.2) for every linear functional $h^* \in B^*$

$$\int_0^1\Big|\langle h^*, f_n(t)\rangle\Big|dt = \frac{1}{2^n}\int_0^1\Big|\sum_{j=1}^{9^n}r_j(t)\langle h^*, x_{j,n}\rangle\Big|dt \leq \frac{1}{2^n}\Big(\sum_{j=1}^{9^n}|\langle h^*, x_{j,n}\rangle|^2\Big)^{\frac{1}{2}}$$

$$\leq \frac{1}{2^{n-1}}\|h^*\|.$$

This means that $\sum_{n\in\mathbb{N}}\Big|\langle h^*, f_n(t)\rangle\Big|$ is an integrable function and hence $\langle h^*, f_n(t)\rangle$ converge to 0 almost everywhere.

On the other hand, by (3.1) $\|f_n(t)\| \overset{\text{a.e.}}{\geq} \frac{3^n}{2^n}\to\infty$. This completes the proof of Theorem 3.3.

Now we turn our attention to Theorem 3.4. First note that Proposition 3.2 provides one direction of the proof of Theorem 3.4. To establish the converse let us introduce some notation.

Let $L = \{(n,k) : n \in \mathbb{N}\cup\{0\}, k = 1,2,\dots,2^n\}$ and recall that $\{x_{n,k} : (n,k) \in L\} \subset B$ is a tree in B provided $x_{n,k} = (1/2)(x_{n+1,2k-1} + x_{n+1,2k})$ for each $(n,k) \in L$. In the future $x_{0,1}$ will always be assumed to be 0. A tree $\langle x_{n,k}\rangle$ is bounded provided $\sup_{(n,k)\in L}\|x_{n,k}\| < \infty$

A tree in B can also be considered as a sequence of B-valued functions defined on the interval $[0,1)$. Let $F_{n,k} = [(k-1)/2^n, k/2^n) \subset [0,1)$, let Σ_n be the algebra generated by sets $F_{n,k} : 1 \leq k \leq 2^n$,

$$X_n(t) = \sum_{k=1}^{2^n} x_{n,k}\chi_{F_{n,k}},$$

and $Y_n = X_{n+1} - X_n$. Clearly, $X_0(t) \equiv 0$. Observe that if $x^* \in B^*$ then $x^*Y_n \in L_2([0,1),\Sigma_{n+1})$ and $\int_G x^*Y_n = 0$ for all $G \in \Sigma_n$. It follows that x^*Y_n is orthogonal to all Σ_n- measurable functions in L_2 and hence, by the Pythagorean theorem,

$$\|x^*X_n\|^2_{L_2} = \sum_{j=0}^{n-1}\|x^*Y_j\|^2_{L_2}.$$

Also recall that if $\langle X_n\rangle$ is a sequence of B-valued functions defined on $[0,1]$, each X_n being Σ_n-measurable, such that $\int_G(X_{n+1} - X_n) = 0$ for all $G \in \Sigma_n$, then the values $x_{n,k} = X_n\Big(\frac{k-\frac{1}{2}}{2^n}\Big)$ form a tree in B.

Let $X = (X_n)$ be a tree in B with $Y_n = X_{n+1} - X_n$ as its sequence of differences and $X^* = (X^*_n)$ a sequence of B^*-valued Σ_n- measurable functions. The pair (X, X^*) is said to be a coherent system in B if:

1. There is an $\varepsilon > 0$ such that for every point $t \in [0,1]$ and for every $n \leq m$

$$\langle X^*_m(t), X_n(t)\rangle \geq n\varepsilon\,.$$

2. The sets $\{\|X^*_n(t)\| : n \in \mathbb{N}, t \in [0,1]\}$ and $\{\|Y_n(t)\| : n \in \mathbb{N}, t \in [0,1]\}$ are each bounded.

Theorem 3.5. ([1]) *Let B be a separable Banach space with a nonseparable dual B^*, then B contains a coherent system.*

Further we show that the sequence $f_n(t) = \frac{1}{n} \cdot X_n(t)$ will be bounded, weakly almost everywhere convergent to 0 but it is not weakly convergent to 0 at any point.

Suppose $\|X_n^*(t)\| \le C_1$ and $\|Y_n(t)\| \le C_2$ for all points $t \in [0,1]$, where C_1 and C_2 are some positive reals. Then

$$\|f_n(t)\| \le \frac{1}{n} \sum_{j=0}^{n-1} \|Y_j(t)\| \le C_2.$$

Now fix some point $t_0 \in [0,1]$. Observe that if $m < n$ then $\langle X_n^*(t_0)\,,\, f_m(t_0)\rangle \ge \varepsilon$. Let x^* be a weak*-limit point of $\langle X_n^*(t_0)\rangle$, and note that, for each $m \in \mathbb{N}$, $x^*(f_m(t_0)) \ge \varepsilon$. It follows that $f_n(t)$ is not weakly convergent to 0 at any point $t \in [0,1]$.

It therefore remains only to proof convergence weakly almost everywhere. Consider an arbitrary functional $x^* \in X^*$. By A denote the set of all points $t \in [0,1]$, such that $x^* \circ f_n(t)$ doesn't converge to 0. We must check that $\mu(A) = 0$. It was shown in [3, p. 30–31] that for and any $h > 0$ and any natural n

$$\mu\Big\{t : \Big|x^* \circ X_n(t)\Big| > h\Big\} \le 2\exp\Big(-\frac{h^2}{2a^2}\Big) \tag{3.3}$$

where $a = \Big(\sum\limits_{j=0}^{n-1} \|x^* \circ Y_j\|_\infty^2\Big)^{\frac{1}{2}}.$

For convenience set $C = C_2\|x^*\|$. Then $a^2 = \sum\limits_{j=0}^{n-1} \|x^* \circ Y_j\|_\infty^2 \le C^2 \cdot n$. Substituting $n^{\frac{3}{4}}$ for h in (3.3), we get

$$\mu\Big(\Big\{t : \Big|\langle x^*\,,\, \frac{1}{n} X_n(t)\rangle\Big| > \frac{1}{n^{\frac{1}{4}}}\Big\}\Big) \le 2\exp\Big(-\frac{\sqrt{n}}{2C^2}\Big).$$

Evidently for any natural m

$$A \subset \bigcup_{n=m}^{\infty} \Big\{t : |x^* \circ f_n(t)| > \frac{1}{n^{\frac{1}{4}}}\Big\}.$$

Therefore

$$\mu(A) \le \sum_{n=m}^{\infty} \mu\Big(\Big\{t : |x^* \circ f_n(t)| > \frac{1}{n^{\frac{1}{4}}}\Big\}\Big) \le 2\sum_{n=m}^{\infty} \exp\Big(-\frac{\sqrt{n}}{2C^2}\Big).$$

To conclude the proof, it remains to note that as series $\sum\limits_{n=1}^{\infty} \exp\Big(-\frac{\sqrt{n}}{2C^2}\Big)$ is convergent, it's remainder tends to 0.

References

[1] *J. Connor, M. Ganichev, and V. Kadets*, A characterization of Banach spaces with separable duals via weak statistical convergence. — *J. Math. Anal. Appl.* (2000), v. 244, p. 251–261.

[2] *V.M. Kadets and M.I. Kadets*, Series in Banach spaces. Birkhäuser, Basel, Boston, Berlin (1997).

[3] *M. Ledoux and M. Talagrand*, Probability in Banach spaces. Springer–Verlag, Berlin, Heidelberg, New-York (1980).

[4] *J.R. Shoenfield*, Martin's Axiom. — *Am. Math. Monthly* (1975), v. 82, p. 610–617.

Слабая топология и свойства, которые выполняются почти всюду

В. Кадец, Т. Кучеренко

Пусть B — банахово пространство. Последовательность B-значных функций $\langle f_n \rangle$ слабо почти всюду сходится к 0, если $x^* \circ f_n$ почти всюду сходится к 0 для каждого непрерывного линейного x^* на B. Банахово пространство конечномерно тогда и только тогда, когда каждая слабо почти всюду сходящаяся к 0 последовательность B-значных функций почти всюду ограничена. Если B — сепарабельно, B^* является сепарабельным тогда и только тогда, когда каждая слабо почти всюду сходящаяся к 0 и почти всюду ограниченная последовательность B-значных функций слабо сходится к 0 почти всюду.

Слабка топологія і властивості, які виконуються майже всюди

В. Кадець, Т. Кучеренко

Нехай B — банахів простір. Послідовність B-значних функцій $\langle f_n \rangle$ слабо майже скрізь збігається до 0, якщо $x^* \circ f_n$ майже скрізь збігається до 0 для кожного безперервного лінійного x^* на B. Банахів простір скінченновимірний тоді і тільки тоді, коли кожна послідовність B-значних функцій, що слабо майже скрізь збігається, майже скрізь обмежена. Якщо B — сепарабельний, B^* є сепарабельним тоді і тільки тоді, коли кожна послідовність B-значних функцій, що слабо майже скрізь збігається до 0 і майже скрізь обмежена, слабо збігається до 0 майже скрізь.