SECTORIAL OPERATORS AND INTERPOLATION THEORY

N. J. KALTON AND T. KUCHERENKO

To Michael Cwikel on his 59th. birthday

Abstract. We present a survey of recent applications of interpolation ideas in the study of sectorial operators.

1. Introduction

Our aim is to illustrate some recent applications of interpolation theory in the study of sectorial operator and semigroups. A good general reference for the use of interpolation in this area is the recent book of Haase [16].

It is a well-established principle if A is a sectorial operator with domain $\mathrm{Dom}(A)$ then the properties of A will improve when A is considered as an operator on the real interpolation spaces $(X, \mathrm{Dom}(A))_{\theta,p}$ where $0 < \theta < 1$ and $1 \le p \le \infty$. Here, one could also replace $\mathrm{Dom}(A)$ by the space $\mathcal{D}(A)$ which is defined to the completion of $\mathrm{Dom}(A)$ under the norm $x \to \|Ax\|$ (we assume all sectorial operators are one-one). This idea goes back to Berens and Butzer [4] and Da Prato and Grisvard [12] (see also Lunardi [23]). It is particularly useful when A is a differential operator of some type and the interpolation spaces can be identified as Sobolev spaces or Besov spaces. A typical recent result is that of Dore [13] who shows that if A is invertible then A has an $H^\infty-$calculus on the interpolation space $(\mathrm{Dom}(A), X)_{\theta,p}$.

At the same time both real and complex interpolation methods have been used to test for the $H^\infty-$calculus ([3]) in Hilbert spaces and recently in [21] a new method of interpolation (which was first mentioned by Peetre) was employed to extend their results to arbitrary Banach spaces.

We will also discuss some recent work of the authors on sectorial operators which have a very strong from of $H^\infty-$calculus, which we call an absolute functional calculus. We show that this concept is closely related to real methods of interpolation, possible more general than the (θ, p)-methods, and that for operators with such a functional calculus one can often prove more satisfying general results.

2. Sectorial operators

Let X be a complex Banach space and let A be a closed operator on X. A is called *sectorial* if A has dense domain $\mathrm{Dom}(A)$ and dense range $\mathrm{Ran}(A) = \mathrm{Dom}(A^{-1})$ and for some $0 < \phi < \pi$ the resolvent $(\lambda - A)^{-1}$ is bounded for $|\arg \lambda| \ge \phi$ and satisfies the estimate

$$\sup_{|\arg \lambda| \ge \phi} \|\lambda(\lambda - A)^{-1}\| < \infty.$$

The infimum of such angles ϕ is denoted $\omega(A)$. Also, we use $Sp(A)$ for the spectrum of A.

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Using contour integration we can define a functional calculus for A as follows. Let Σ_ϕ be the open sector $\{z \neq 0 : \ |\arg z| < \phi\}$. If $f \in H^\infty(\Sigma_\phi)$ we say that $f \in H_0^\infty(\Sigma_\phi)$ if there exists $\delta > 0$ such that $|f(z)| \le C \max(|z|^\delta, |z|^{-\delta})$. For $f \in H_0^\infty(\Sigma_\phi)$ where $\phi > \omega(A)$ we can define $f(A)$ by a contour integral, which converges as a Bochner integral in the space of bounded operators on X.

$$f(A) = \frac{1}{2\pi i}\int_{\Gamma_\nu} f(\lambda)(\lambda - A)^{-1} d\lambda$$

where Γ_ν is the contour $\{|t|e^{-i\nu \mathrm{sgn}\, t} : \ -\infty < t < \infty\}$ and $\omega(A) < \nu < \phi$. We can then estimate $\|f(A)\|$ by

$$\|f(A)\| \le C_\phi \int_{\Gamma_\nu} |f(\lambda)| \frac{|d\lambda|}{|\lambda|}.$$

If we have a stronger estimate

$$\|f(A)\| \le C\|f\|_{H^\infty(\Sigma_\phi)} \qquad f \in H_0^\infty(\Sigma_\phi)$$

then we say that A *has an* $H^\infty(\Sigma_\phi)-$*calculus*; in this case we may extend the functional calculus to define $f(A)$ for every $f \in H^\infty(\Sigma_\phi)$. The infimum of all such angles ϕ is denoted by $\omega_H(A)$. The study of sectorial operators with an H^∞-calculus was initiated by McIntosh [24].

In the general case it possible to extend the functional calculus only to certain $f \in H^\infty(\Sigma_\phi)$. In fact if $x \in \mathrm{Dom}(A) \cap \mathrm{Ran}(A)$ the Bochner integral

$$f(A)x = \frac{1}{2\pi i}\int_{\Gamma_\nu} f(\lambda)(\lambda - A)^{-1} x\, d\lambda$$

converges and allows to define $f(A)$ densely. If $f(A)$ extends to a bounded operator on X then we say f belongs to the functional calculus of A.

To motivate the above definitions it is useful to consider the example of differentiation D on the spaces $L_p(\mathbb{R})$ for $1 \le p < \infty$. Here

$$\mathrm{Dom}(D) = \{f \in L_p : \ f \text{ is absolutely continuous and } f' \in L_p\}.$$

Then D is sectorial with $\omega(D) = \pi/2$; if $1 < p < \infty$ D has an $H^\infty-$calculus and $\omega_H(D) = \pi/2$. For a full description of sectorial operators we refer to [10].

At this point let us also introduce the notion of *R-boundedness* of families of operators. If $\mathcal{T}$ is a family of bounded operators on a Banach space X then $\mathcal{T}$ is said to be *R-bounded* if there is a constant C so that we have

$$(\mathbb{E}\|\sum_{j=1}^n \epsilon_j T_j x_j\|^2)^{1/2} \le C(\mathbb{E}\|\sum_{j=1}^n \epsilon_j x_j\|^2)^{1/2}$$

whenever $x_1, \ldots, x_n \in X$ and $T_1, \ldots, T_n \in \mathcal{T}$. Here $\epsilon_1, \ldots, \epsilon_n$ denotes a sequence of independent Rademachers. If X is a Hilbert space this condition reduces to boundedness of the collection $\mathcal{T}$ but in general it is a stronger condition (cf. [17]).

The concept of R-boundedness which is implicitly due to Bourgain [6] and was later formalized and studied by Berkson and Gillespie [5] and Clément, de Pagter, Sukochev and Witvliet [9]. It has proved especially important in the theory of sectorial operators. A sectorial operator A is called *R-sectorial* if there exists $\omega(A) < \phi < \pi$ so that $\{\lambda(\lambda - A)^{-1} : \arg\lambda > \phi\}$ is an R-bounded family. The infimum of all such angles is denoted $\omega_R(A)$. A is called *almost R-sectorial* if for some $\omega(A) < \phi < \pi$ we have $\{\lambda A(\lambda - A)^{-2} : \ \arg\lambda > \phi\}$ is an R-bounded family. The infimum of all such angles is denoted $\tilde{\omega}_R(A)$.

3. The Joint Functional Calculus

In this section we will discuss the joint functional calculus of two commuting sectorial operators.

Suppose we have two sectorial operators A and B on X which commute (i.e. their resolvents commute). Then $\mathrm{Dom}(A) \cap \mathrm{Dom}(B)$ is a dense subspace of X. We can then define a *joint functional calculus for* (A, B) via a similar procedure. In this case we suppose $\phi_A > \nu_A > \omega(A)$, $\phi_B > \nu_B > \omega(B)$ and then that $f \in H_0^\infty(\Sigma_{\phi_A} \times \Sigma_{\phi_B})$ i.e that f is analytic on the product of the sectors Σ_{ϕ_A} and Σ_{ϕ_B} and satisfies an estimate

$$|f(w,z)| \leq C \max(|w|^\delta, |w|^{-\delta}) \max(|z|^\delta, |z|^{-\delta}).$$

Then $f(A, B)$ can be defined by

$$f(A,B) = \frac{-1}{4\pi^2} \int_{\Gamma_{\nu_A}} \int_{\Gamma_{\nu_B}} f(\lambda,\mu)(\lambda - A)^{-1}(\mu - B)^{-1} d\lambda \, d\mu.$$

As before we say that (A, B) has a *joint* $H^\infty(\Sigma_{\phi_A} \times \Sigma_{\phi_B})$*-functional calculus* if we have an estimate

$$\|f(A,B)\| \leq C\|f\|_{H^\infty(\Sigma_{\phi_A} \times \Sigma_{\phi_B})}.$$

In general we say $f \in H^\infty(\Sigma_{\phi_A} \times \Sigma_{\phi_B})$ *belongs to the joint functional calculus of* (A, B) provided $f(A, B)$ (defined on $\mathrm{Dom}(A) \cap \mathrm{Ran}(A) \cap \mathrm{Dom}(B) \cap \mathrm{Ran}(B)$) extends to a bounded operator on X.

We describe a typical problem in the area. Let us suppose A and B are two commuting sectorial operators on a Banach space X. Then $\mathrm{Dom}(A) \cap \mathrm{Dom}(B)$ is a dense subspace of X and thus $A + B$ is a densely defined operator, which is in fact closable; that is we can define a closed operator $A + B$ on $\mathrm{Dom}(A + B) \supset \mathrm{Dom}(A) \cap \mathrm{Dom}(B)$. However, for many applications we need to know that $A+B$ is already closed on $\mathrm{Dom}(A) \cap \mathrm{Dom}(B)$ i.e. $\mathrm{Dom}(A) \cap \mathrm{Dom}(B) = \mathrm{Dom}(A+B)$. If we assume that $A+B$ is invertible so that the equation $(A + B)x = y$ always has a solution for $y \in X$ then $\mathrm{Dom}(A) \cap \mathrm{Dom}(B) = \mathrm{Dom}(A + B)$ if and only if we have an estimate

$$\|Ax\| \leq C\|Ax + Bx\| \qquad x \in \mathrm{Dom}(A) \cap \mathrm{Dom}(B).$$

This is equivalent to requiring that $f(A, B)$ is bounded when $f(w, z) = w(w + z)^{-1}$. If we assume that $\omega(A) + \omega(B) < \pi$ then we can find $\phi > \omega(A)$, $\psi > \omega(B)$ so that $f \in H^\infty(\Sigma_\phi \times \Sigma_\psi)$.

A special and important case is the Cauchy problem

$$u'(t) + B_0 u(t) = f(t) \qquad 0 \leq t \leq 1$$

with initial condition $u(0) = 0$. Here B_0 is a sectorial operator on a Banach space X with $\omega(B_0) < \pi/2$ and we consider the problem in a suitable $X-$valued function space e.g. $L_p([0, 1]; X)$ for $1 < p < \infty$ or $\mathcal{C}([0, 1]; X)$. In this setting we take $A = d/dt$ and B to be the sectorial operator induced by B_0 on the function space.

In 1987, Dore and Venni [15] proved a very useful general result that $\mathrm{Dom}(A + B) = \mathrm{Dom}(A) \cap \mathrm{Dom}(B)$ which required the hypotheses that X be a UMD-space and that A, B both have *bounded imaginary powers*. An alternative approach was developed in [22] under the stronger assumption on A that it has an $H^\infty-$calculus but with less stringent conditions on B and X:

Theorem 3.1. *Let A, B be commuting sectorial operators on a Banach space X such that A has an H^∞-calculus. Suppose $\phi > \omega_H(A)$, $\psi > \omega(A)$ and suppose $f \in H^\infty(\Sigma_\phi \times \Sigma_\psi)$. Then* *for $f(A,B)$ to be a bounded operator it suffices that for each $z\in\Sigma_\psi$, $f(z,B)$ is bounded and the collection $\{f(z,B):\ z\in\Sigma_\phi\}$ is R-bounded.*

In particular if $\omega_H(A)+\omega(B)<\pi$ if we take $f(w,z)=w(w+z)^{-1}$ we see that $f(A,B)$ is bounded provided B is R-sectorial and $\omega_H(A)+\omega_R(B)<\pi$. This result recovers a theorem of Weis [25] on L_p-maximal regularity.

4. Interpolation and the H^∞-calculus

Suppose A is a sectorial operator on X. We define an ambient space Y to be the completion of X under the norm $x\to\|A(1+A)^{-2}x\|$. Then A,A^{-1} can be extended to *bounded* operators $A,A^{-1}:X\to Y$. We then consider two subspaces of Y

$$\mathcal{D}(A)=\{A^{-1}x: x\in X\},\qquad \mathcal{R}(A)=\{Ax:x\in X\}.$$

Then $\mathcal{D}(A)$ is a Banach space under the norm $x\mapsto\|Ax\|$ and $\mathcal{R}(A)$ is a Banach space under the norm $x\mapsto\|A^{-1}x\|$. Both these spaces are densely embedded into Y. Note that $\mathcal{R}(A)=\mathcal{D}(A^{-1})$. It is also possible to define $\mathcal{D}(A^a)$ for any real a and we will later have need of these spaces.

We can now consider the Banach couples $(\mathcal{D}(A),\mathcal{R}(A))$ or $(\mathcal{D}(A),X)$. If we compute the K-functionals we have

$$K(t,x;\mathcal{D}(A),X)=\|tA(t+A)^{-1}x\|\qquad x\in X$$

and

$$K(t,x;\mathcal{D}(A),\mathcal{R}(A))=\|tA(t+A)^{-2}x\|\qquad x\in X.$$

In [24] McIntosh showed that when X is a Hilbert space then A has an $H^\infty-$calculus for some angle if and only if

$$\|x\|\approx\left(\int_0^\infty\|tA(t+A)^{-2}x\|^2\frac{dt}{t}\right)^{1/2}\qquad x\in X.$$

This of course has an interpretation in terms of interpolation, which was exploited in [3]. We note that for interpolation of *Hilbert* spaces the real $(\theta,2)$-method coincides with the complex $\theta-$method of interpolation.

Theorem 4.1. *Let X be a Hilbert space, and let A be a sectorial operator on X. In order that A admits an $H^\infty-$calculus it is necessary and sufficient that $X=(\mathcal{D}(A),\mathcal{R}(A))_{1/2,2}=[\mathcal{D}(A),\mathcal{R}(A)]_{1/2}$.*

Corollary 4.2. *Let X be a Hilbert space, and let A be a sectorial operator on X. Suppose B is another sectorial operator on X so that $\|Ax\|\approx\|Bx\|$ and $\|A^{-1}x\|\approx\|B^{-1}x\|$ for $x\in X$. If A admits an H^∞-calculus, then so does B.*

This Corollary provides a powerful technique for establishing that certain sectorial operators admit an $H^\infty-$calculus. For example $D=d/dt$ admits an $H^\infty-$calculus on $L_2(\mathbb{R})$ and so any sectorial operator which is “suitably similar” to D also admits an $H^\infty-$calculus. It would, of course be nice to have a similar result for arbitrary Banach spaces but this hope is illusory. In fact it is not difficult to find a sectorial operator A with an $H^\infty-$calculus so that $\omega(A)<\omega_H(A)<\pi$. Then if we take $B=e^{i\theta}A$ or $e^{-i\theta}A$ where $\pi-\omega_H(A)<\theta<\pi-\omega(A)$ then one choice of B fails to have an $H^\infty-$calculus, but clearly the hypotheses of Corollary 4.2 hold. Such an A is constructed in [18]. There is, however, one special case where a similar result holds:

Theorem 4.3. *Let $X=L_1(\mu)$ for some measure μ and let A be a sectorial operator on X. Then A admits an $H^\infty-$calculus if and only if $X=(\mathcal{D}(A),\mathcal{R}(A))_{1/2,1}$.*

Theorem 4.3 is proved in [22]; the crucial property of L_1 is that it is a so-called GT-space. For details we refer to [22].

In the general case, it turns out that a result analogous to Theorem 4.1 can be proved but it requires a different interpolation method and an additional hypothesis. We now describe the so-called Rademacher method of interpolation. Let (Y_0, Y_1) be a Banach couple. Then the space $\langle Y_0, Y_1\rangle_\theta$ for $0 < \theta < 1$ consists of all $y \in Y_0 + Y_1$ of the form

$$y = \sum_{n\in\mathbb{Z}} y_n$$

(with convergence in $Y_0 + Y_1$) such that

$$\|y\|_{\langle Y_0,Y_1\rangle_\theta} = \inf\left\{\sup_N \max\left(\left(\mathbb{E}\Big\|\sum_{k=-N}^{N} 2^{-k\theta}\epsilon_k y_k\Big\|_{Y_0}^2\right)^{1/2}, \left(\mathbb{E}\Big\|\sum_{k=-N}^{N} 2^{k(1-\theta)}\epsilon_k y_k\Big\|_{Y_0}^2\right)^{1/2}\right)\right\}$$

where the infimum is taken over all such representations.

The following result is proved in [21]:

Theorem 4.4. *Let X be a Banach space with nontrivial Rademacher type and suppose A is a sectorial operator on X. Then A admits an $H^\infty-$calculus with $\omega_H(A) \le \phi$ if and only if:*
(i) A is almost R-sectorial with $\tilde{\omega}_R(A) \le \phi$, and
(ii) $X = \langle \mathcal{D}(A), \mathcal{R}(A)\rangle_{1/2}$.

In fact a somewhat more technical statement can be made if X fails to have nontrivial type (see [21]). This leads to an analogue of Corollary 2, but under the additional assumption that B is almost R-sectorial. See also [19] for another treatment.

In another direction, Dore [13] and [14] considers an arbitrary sectorial operator A on X and shows that A has an $H^\infty-$calculus when considered as a sectorial operator on the real interpolation spaces $(\mathrm{Dom}(A) \cap \mathrm{Ran}(A), X)_{\theta,p}$ where $0 < \theta < 1$ and $1 \le p < \infty$. When A is invertible (i.e. 0 is in the resolvent set of A) the space $\mathrm{Dom}(A) \cap \mathrm{Ran}(A)$ reduces to $\mathcal{D}(A)$. An analogous result is true when $p = \infty$ except that A is not necessarily sectorial on the interpolation space, because its domain (or range) need not be dense.

5. The absolute functional calculus

If we examine Theorems 4.1 and 4.3 we see that in each case X is described as a real interpolation space between $\mathcal{D}(A)$ and $\mathcal{R}(A)$. There are other results in the literature which suggest that a sectorial operator behaves "better' on a real interpolation space than on the original space (e.g. [12], [13], [14]). Thus we are led to the problem of characterizing sectorial operators for which X is a real interpolation space between $\mathcal{D}(A)$ and $\mathcal{R}(A) = \mathcal{D}(A^{-1})$, or, more generally, $\mathcal{D}(A^{-a})$ and $\mathcal{D}(A^b)$ for some $a, b > 0$.

We shall say that A has an *absolute functional calculus* if for some $0 < C < \infty$, $\phi > \omega(A)$ and $g, h \in H_0^\infty(\Sigma_\phi)$ we have the implication

$$\|g(tA)h(tA)x\| \le C\|g(tA)y\| \quad 0 < t < \infty \implies \|x\| \le C\|y\|.$$

In general we are interested in specific choices for g, h even though some results hold without this restriction. For $a, b > 0$ we define

$$\varphi_{a,b}(z) = \frac{z^a}{(1+z)^{a+b}}.$$

We say that A has an $(a,b)-$*absolute functional calculus* if in the above definition we can take $g=\varphi_{a,b}$ and $h=\varphi_{\delta,\delta}$ for some $\delta>0$.

Let us explain the connection with interpolation. Let (X_0,X_1) be a Banach couple. It follows from the principle of K-divisibility of Brudnyi and Krugljak (see [7], [8] and [11]) that an intermediate space X can be realized as a real interpolation space $(X_0,X_1)_E$ if and only if it satisfies the condition of $K-$monotonicity i.e.

$$K(t,x)\le K(t,y)\quad 0<t<\infty,\ y\in X\qquad\Longrightarrow\ x\in X,\ \|x\|_X\le C\|y\|_X.$$

Here E is an interpolation space between $L_\infty(0,\infty)$ and

$$L^1_\infty(0,\infty)=\{f\in L_0(0,\infty):\ \sup_{t>0}t|f(t)|<\infty\}.$$

(Here $L_0(0,\infty)$ denotes the space of all measurable functions on $(0,\infty)$.)Then $(X_0,X_1)_E$ is the space of $x\in X_0+X_1$ such that $K(t,x)\in E$ with the norm $x\to\|K(t,x)\|_E$.

Let us recall that an interpolation space for (X_0,X_1) is regular if $X_0\cap X_1$ is dense in X. It is easy to show that $(X_0,X_1)_E$ is regular if the space $\mathcal{C}_{00}(0,\infty)$ of continuous functions of compact support are dense in E.

For our purposes it is necessary to impose a stronger condition, which eliminates spaces which are too close to the endpoints X_0 and X_1. We consider the *Boyd indices* of E which we define by

$$\beta_E=\limsup_{s\to\infty}\frac{\log\|D_s\|}{\log s}\qquad\alpha_E=\liminf_{s\to0}\frac{\log\|D_s\|}{\log s}$$

where D_s is the dilation operator on E defined by $D_sf(t)=f(t/s)$. In general we have $-1\le\alpha_E\le\beta_E\le0$. We say that X is a *strict* real interpolation space if it is regular and E can be chosen so that $-1<\alpha_E\le\beta_E<0$. Of course the standard $(\theta,p)-$methods yield strict interpolation spaces with $\alpha_E=\beta_E=-\theta$, when $1\le p<\infty$.

The following theorem (see [20]) modifies the original Brudnyi-Krugljak characterization of real interpolation spaces:

Theorem 5.1. *Let (X_0,X_1) be a Banach couple and let X be a regular interpolation space. Then X is a strict real interpolation space for the pair (X_0,X_1) if there is a constant $0<C<\infty$ and $\delta>0$ so that if $y\in X,\ x\in X_0+X_1$ and $s>0$,*

$$K(t,x)\le K(st,y),\quad 0<t<\infty\implies x\in X,\ \|x\|_X\le C\max(s^{1-\delta},s^{\delta})\|y\|_X.$$

If $a,b\in\mathbb{R}$ and we consider a regular interpolation space Y for the couple $(\mathcal{D}(A^a),\mathcal{D}(A^b))$ then A induces a sectorial operator on Y, which we continue to denote by A.

We are now in a position to state the connection between real interpolation and the absolute functional calculus [20].

Theorem 5.2. *If E defines a regular strict real interpolation method and $a<b\in\mathbb{R}$ then A considered as a sectorial operator on $(\mathcal{D}(A^a),\mathcal{D}(A^b))_E$ has an absolute functional calculus of type $(b-a,b-a)$.*

Theorem 5.3. *If A has an absolute functional calculus of type (a,b) on X then, provided $a'>a,\ b'>b$, X is a strict real interpolation space for the couple $(\mathcal{D}(A^{-a'}),\mathcal{D}(A^{b'}))$.*

These theorems allow us to give many examples of sectorial operators with an absolute functional calculus. If we let $A=I-\Delta$ (where Δ is the Laplacian) then the spaces $\mathcal{D}(A^m)$ for $m\ge1$ are the Sobolev spaces $W^{2m}_q(\mathbb{R}^N;X)$. Thus for $1\le q,r,s<\infty$

$$(L_q(\mathbb{R}^N;X),W^{2m}_q(\mathbb{R}^N;X))_{s/2m,r}=B^s_{q,r}(\mathbb{R}^N;X)$$

is a Besov space [1]. On these spaces A has an absolute functional calculus.

6. Applications of the absolute functional calculus.

Theorem 6.1. *Let A be a sectorial operator with an absolute functional calculus. Then A has an $H^\infty-$calculus with $\omega_H(A) = \omega(A)$.*

Notice here that we have already observed, it is possible to have $\omega_H(A) > \omega(A)$ in general for operators with an $H^\infty-$calculus. We also note that Dore's theorem [13] is a Corollary of this result and Theorem 5.2.

Once A has an absolute functional calculus it is possible to improve Theorem 3.1. One can replace the hypothesis of R-boundedness (which is quite difficult to verify) by simple boundedness:

Theorem 6.2. *Suppose A and B are commuting sectorial operators and A has absolute calculus. Suppose further that $f \in H^\infty(\Sigma_\phi \times \Sigma_\psi)$ where $\phi > \omega(A)$, $\psi > \omega(B)$ is such that $f(z, B)$ is a bounded operator for every $z \in \Sigma_\phi$ and the family $\{f(z, B) : z \in \Sigma_\phi\}$ is uniformly bounded. Then $f(A, B)$ is a bounded operator.*

Now let us return to the discussion of the equation $Ax + Bx = y$ where A and B are commuting sectorial operators. We have immediately from Theorem 6.2 the following:

Theorem 6.3. *Let A, B be commuting sectorial operators such that A has an absolute functional calculus. If $\omega(A) + \omega(B) < \pi$ then $(A + B, Dom(A) \cap Dom(B))$ is a closed operator.*

Note here the distinction from merely assuming that A has an $H^\infty-$calculus. In that case it is necessary to add the hypothesis that $\omega_H(A) + \omega_R(B) < \pi$ [22].

Inspired by a recent paper of Arendt, Batty and Bu [2] we turn the consideration of conditions when the closure of $A+B$ (i.e. $(A+B, \mathrm{Dom}(A+B))$ is invertible, i.e. $(A+B)^{-1}$ extends to a bounded operator. Here we merely assume that B is a closed operator with non-empty resolvent set.

Theorem 6.4. *Suppose A is a sectorial operator with an absolute functional calculus such that $-A$ is the generator of a bounded group. Suppose B is an invertible closed operator which commutes with A. Assume further that $Sp(B) \cap i\mathbb{R} = \emptyset$. Then each of the following conditions implies that $(A + B, Dom(A + B))$ is invertible:*
(i) For some $\phi < \pi/2$ we have $Sp(B) \subset \Sigma_\phi \cup (-\Sigma_\phi)$ and

$$\sup_{\lambda \notin \Sigma_\phi \cup (-\Sigma_\phi)} \|(\lambda + B)^{-1}\| < \infty.$$

(ii) We have:

$$\sup_{-\infty < t < \infty} |t|^{\frac{1}{2}} \|(it + B)^{-1}\| < \infty.$$

There is a periodic case of the same theorem:

Theorem 6.5. *Suppose A is a sectorial operator with an absolute functional calculus such that $-A$ is the generator of a bounded group with $e^{-2\pi A} = I$. Suppose B is a closed operator which commutes with A. Assume further that $Sp(B) \cap i\mathbb{Z} = \emptyset$. Then the following condition implies that $(A + B, Dom(A + B))$ is invertible:*

$$\sup_{k \in \mathbb{Z}} |k|^{\frac{1}{2}} \|(ik + B)^{-1}\| < \infty.$$

Let us now consider a special case of Theorem 6.5 considered by Arendt, Batty and Bu [2]:

$$u'(t) + Bu(t) = f(t) \qquad u(0) = u(2\pi)$$

in the space $\mathcal{C}^\alpha(\mathbb{T}; X)$ of $2\pi-$periodic α-Hölder continuous functions $u : \mathbb{R} \to X$. We consider this as a problem on $\mathbb{R}/2\pi\mathbb{Z} = \mathbb{T}$. The problem is to determine conditions on B which guarantee the existence of mild solutions in $\mathcal{C}^\alpha(\mathbb{T}; X)$ for this equation for every $f \in \mathcal{C}^\alpha(\mathbb{T}; X)$ with mean zero.

It does not essentially change the problem to use the "little" Hölder space $\mathcal{C}^{\alpha,0}(\mathbb{T}; X)$ of all functions $u \in C^\alpha(\mathbb{T}; X)$ such that

$$\lim_{|t-s|\to 0} \frac{\|u(t) - u(s)\|}{|t-s|^\alpha} = 0.$$

We can also restrict to those functions of mean zero $\mathcal{C}_0^{\alpha,0}(\mathbb{T}; X)$.

Let $\mathcal{C}_0(\mathbb{T}; X)$ be the subspace of $\mathcal{C}(\mathbb{T}; X)$ of all functions of mean zero; then $A = d/dt$ on $\mathcal{C}_0(\mathbb{T}; X)$ then A is sectorial. Then A is also sectorial and will have an absolute functional calculus on the inner interpolation space $(\mathcal{C}_0(\mathbb{T}; X), \mathcal{D}(A))_{(\alpha,\infty)}$ which is defined to be the closure of $\mathcal{C}_0(\mathbb{T}; X) \cap \mathcal{D}(A)$ in the interpolation space $(\mathcal{C}_0(\mathbb{T}; X), \mathcal{D}(A))_{\alpha,\infty}$. This space now coincides with $\mathcal{C}_0^{\alpha,0}(\mathbb{T}; X)$. Therefore our equation becomes of the form

$$(A + \tilde{B})u = f$$

where A has an absolute functional calculus, and $\tilde{B}$ is the closed operator induced on $\mathcal{C}_0^{\alpha,0}(\mathbb{T}; X)$ by B. Thus Theorem 6.5 can be applied directly and we see that

$$\sup_{k\in\mathbb{Z}} |k|^{\frac{1}{2}} \|(ik + B)^{-1}\| < \infty$$

is a *sufficient* condition for the invertibility of $A + \tilde{B}$. This result was obtained in [2] under the extra hypothesis that X has nontrivial Rademacher type; the techniques used in [2] relied on Fourier multiplier theorems.

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Department of Mathematics, University of Missouri-Columbia, Columbia, MO 65211
E-mail address: `nigel@math.missouri.edu`

Department of Mathematics, University of California-Los Angeles, Los Angeles, CA 90095
E-mail address: `tamara@math.ucla.edu`