R-BOUNDED APPROXIMATING SEQUENCES AND APPLICATIONS TO SEMIGROUPS

M. HOFFMANN, N. KALTON AND T. KUCHERENKO

Abstract. It is shown that on certain Banach spaces, including $C[0,1]$ and $L_1[0,1]$, there is no strongly continuous semigroup $(T_t)_{0<t<1}$ consisting of weakly compact operators such that $(T_t)_{0<t<1}$ is an R-bounded family. More general results concerning approximating sequences are included and some variants of R-boundedness are also discussed.

1. Introduction

Recent work on semigroup theory ([13], [24]) has highlighted the importance of the concept of R-boundedness . Let us recall the definition of R-bounded families of operators (cf. [2], [9], [7]).

Definition 1.1. A family $\mathcal{T}$ of operators in $\mathcal{L}(X,Y)$ is called *R-bounded* with R-boundedness constant $C>0$ if letting $(\epsilon_k)_{k=1}^\infty$ be a sequence of independent Rademachers on some probability space then for every $x_1,\ldots,x_n\in X$ and $T_1,\ldots,T_n\in\mathcal{T}$ we have

$$\left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k T_k x_k\right\|^2\right)^{\frac12} \le C\left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k x_k\right\|^2\right)^{\frac12}. \tag{1.1}$$

By the Kahane-Khintchine inequality we can replace 2 above by any other exponent $1\le p<\infty$ to obtain an equivalent definition. We will also need the following definition introduced in [13].

Definition 1.2. A family $\mathcal{T}$ of operators in $\mathcal{L}(X,Y)$ is called *WR-bounded* with WR-boundedness constant $C>0$ if for every $x_1,\ldots,x_n\in X, y_1^*,\ldots,y_n^*\in Y^*$ and $T_1,\ldots,T_n\in\mathcal{T}$ we have

$$\sum_{k=1}^n |\langle T_k x_k, y_k^*\rangle| \le C\left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k x_k\right\|^2\right)^{\frac12}\left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k y_k^*\right\|^2\right)^{\frac12}. \tag{1.2}$$

It is clear by the Cauchy-Schwarz inequality that R-boundedness implies WR-boundedness. The converse is not true in general, but it holds for spaces with non-trivial type ([20], [13]).

In [13] it was shown that no reasonable differential operator on L_1 can have an $H^\infty-$calculus. In this note we consider the related question whether a differential-type operator on L_1 can generate an R-bounded semigroup. Note that if A is an

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R-sectorial operator (cf. [13]) with R-sectoriality angle less than $\frac{\pi}{2}$ then the semigroup $(e^{-tA})_{0<t<1}$ is necessarily R-bounded. In general, one expects a semigroup generated by a differential operator on a bounded domain to consist of weakly compact operators. We are thus led to consider the question whether one can have a strongly continuous semigroup $(T_t)_{0<t<1}$ on L_1 such that each T_t is weakly compact (or equivalently compact, since L_1 has the Dunford-Pettis property) and such that the family $(T_t)_{0<t<1}$ is R-bounded. In fact this leads to considering versions of the approximation property; the only property of the semigroup needed is commutativity. We consider the general question whether on a given separable Banach space one can find an R-bounded sequence $(T_n)_{n\in\mathbb{N}}$ of commuting weakly compact operators such that $\lim_{n\to\infty} T_n x = x$ for all $x \in X$. Our main results show that for the spaces $L_1[0,1]$, $C(K)$ (except c_0) and the disk algebra $A(\mathbb{D})$ this is impossible. These results may be regarded as extensions of classical results that the spaces $L_1, C(K)$ do not have unconditional bases.

In the case of L_1 we are led to consider a natural weakening of R-boundedness, where we use the definition (1.1) but only for single vectors.

Definition 1.3. A family $\mathcal{T}$ of operators in $\mathcal{L}(X,Y)$ is called *semi-R-bounded* if there is a constant $C>0$ such that for every $x \in X, a_1,\dots,a_n \in \mathbb{C}$ and $T_1,\dots,T_n \in \mathcal{T}$ we have

$$\left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k a_k T_k x\right\|^2\right)^{\frac{1}{2}} \le C\left(\sum_{k=1}^n |a_k|^2\right)^{\frac{1}{2}} \|x\|. \tag{1.3}$$

We note that semi-R-boundedness is equivalent to R-boundedness for operators on L_1. In Theorem 2.2 we actually characterize all spaces where semi-R-boundedness is equivalent to R-boundedness as spaces which are either Hilbert spaces or GT-spaces of cotype 2 in the terminology of Pisier [19].

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2. R-boundedness and WR-boundedness

In this section, we make some remarks about R-boundedness and related notions.

Note that in a space of type 2, any uniformly bounded collection $\mathcal{T} \subset L(X,X)$ is semi-R-bounded. The converse is also true:

Proposition 2.1. *A Banach space X has type 2 if and only if uniform boundedness is equivalent to semi-R-boundedness.*

Proof. Suppose that every uniformly bounded family of operators is already semi-R-bounded. Pick any $x \in X$ and $x^* \in X^*$ such that $\|x\| = \|x^*\| = 1$ and $x^*(x) = 1$. Notice that the family $\mathcal{T} = \{x^* \otimes u : \|u\| = 1\}$ is uniformly bounded with constant one and hence semi-R-bounded by assumption. Let C be the semi-R-boundedness constant of $\mathcal{T}$. Select any $x_1,\dots,x_n \in X$ and write $x_k = \|x_k\|u_k$ where $\|u_k\| = 1$. Then $\{x^* \otimes u_k : k = 1,\dots,n\} \subset \mathcal{T}$ and

$$
\begin{aligned}
\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k x_k\right\|^2\right)^{\frac{1}{2}} &= \left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k \|x_k\|(x^*\otimes u_k)x\right\|^2\right)^{\frac{1}{2}} \\
&\le C\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k \|x_k\| x\right\|^2\right)^{\frac{1}{2}} \\
&= C\|x\|\left(\sum_{k=1}^{n}\|x_k\|^2\right)^{\frac{1}{2}} \\
&= C\left(\sum_{k=1}^{n}\|x_k\|^2\right)^{\frac{1}{2}}
\end{aligned}
$$

Thus, X has type 2. □

For some spaces, semi-R-boundedness is equivalent to R-boundedness and we are able to completely characterize these spaces in the next theorem. Let us recall that a Banach space X is called a *GT-space* if every bounded operator $T : X \to \ell_2$ is absolutely summing. Examples of GT-spaces of cotype 2 are L_1, the quotient of L_1 by a reflexive subspace [19],[14], and L_1/H_1 [8]. It is unknown whether every GT-space has cotype 2.

Theorem 2.2. *Suppose X is separable. Then the following are equivalent :*

(i) *Every semi-R-bounded family of operators on X is R-bounded.*
(ii) *X is isomorphic to ℓ_2 or X is a GT-space of cotype 2.*

Proof. First we prove that (i) implies (ii). Suppose that every semi-R-bounded family of operators on X is R-bounded. Let us note that this implies the existence of a constant K so that if $\mathcal{T}$ has semi-R-boundedness constant C then it has R-boundedness constant KC; for otherwise we could find a sequence $\mathcal{T}_n$ of families with semi-R-boundedness constant 1 and R-boundedness constant at least 4^n; then the family $\cup_{n\ge1}2^{-n}\mathcal{T}_n$ contradicts our assumption. Fix $M > 1$ and take $x \in X$. Choose $n \in \mathbb{N}$. By Dvoretzky's theorem [16] we can find $e_1, \dots, e_n \in X$ such that for any $a_1, \dots, a_n \in \mathbb{C}$ we have

$$
M^{-1}\left(\sum_{k=1}^{n}|a_k|^2\right)^{\frac{1}{2}} \le \left\|\sum_{k=1}^{n}a_k e_k\right\| \le M\left(\sum_{k=1}^{n}|a_k|^2\right)^{\frac{1}{2}}.
$$

Consider the family of operators $\mathcal{T}_n = \{u^*\otimes e_k : \|u^*\| = 1, k = 1, \dots, n\}$. Then each $\mathcal{T}_n$ is semi-R-bounded with constant M as follows. A finite subfamily of $\mathcal{T}_n$ is of the form $\{u^*_{kj}\oplus e_k : 1 \le k \le n, 1 \le j \le m_k\}$ for some $m_1, \dots, m_n \in \mathbb{N}$. Then for every $a_{11},\dots,a_{nm_n}\in\mathbb{C}$ we have, (letting ϵ_{kj} denote independent Rademachers),

$$\left(\mathbb{E}\left\|\sum_{k=1}^{n}\sum_{j=1}^{m_k}\epsilon_{kj}a_{kj}u_{kj}^*(x)e_k\right\|^2\right)^{\frac12}$$
$$\le M\left(\mathbb{E}\sum_{k=1}^{n}\left|\sum_{j=1}^{m_k}\epsilon_{kj}u_{kj}^*(x)a_{kj}\right|^2\right)^{\frac12}$$
$$\le M\left(\sum_{k=1}^{n}\mathbb{E}\left|\sum_{j=1}^{m_k}\epsilon_{kj}u_{kj}^*(x)a_{kj}\right|^2\right)^{\frac12}$$
$$\le M\left(\sum_{k=1}^{n}\sum_{j=1}^{m_k}|a_{kj}|^2\right)^{\frac12}\|x\|$$

Our assumption implies that each $\mathcal{T}_n$ is R-bounded with constant KM. Let $x_1,\dots,x_n\in X$ and write $x_k=\|x_k\|u_k$ where $\|u_k\|=1$. Choose $u_k^*\in X^*$ such that $u_k^*(u_k)=1$ and $\|u_k^*\|=1$. Now we have

$$\left(\sum_{k=1}^{n}\|x_k\|^2\right)^{\frac12}\le M\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k\|x_k\|e_k\right\|^2\right)^{\frac12}$$
$$=M\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k\|x_k\|u_k^*(u_k)e_k\right\|^2\right)^{\frac12}$$
$$=M\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k\|x_k\|(u_k^*\otimes e_k)(u_k)\right\|^2\right)^{\frac12}$$
$$\le KM^2\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k\|x_k\|u_k\right\|^2\right)^{\frac12}$$
$$=KM^2\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_kx_k\right\|^2\right)^{\frac12}$$

This shows that X has cotype 2.

Let us assume that X has non-trivial type. Then by results of Pisier [19] and also by Figiel and Tomczak-Jaegermann [12], ℓ_2^n is uniformly complemented in X. Thus, for some constant C, for every $n\in\mathbb{N}$ we can choose a biorthogonal system $\{(e_k,e_k^*):k=1,\dots,n\}$ in $X\times X^*$ such that

$$\left\|\sum_{k=1}^{n}a_ke_k\right\|\le C\left(\sum_{k=1}^{n}|a_k|^2\right)^{\frac12}$$

and

$$\left\|\sum_{k=1}^{n} a_k e_k^*\right\| \leq C\left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}}$$

for all $a_1, \ldots, a_n \in \mathbb{C}$. Note that for any $x \in X$ and $a_1, \ldots, a_n \in \mathbb{C}$,

$$\left|\sum_{k=1}^{n} a_k e_k^*(x)\right| \leq C\left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}} \|x\|$$

and so

$$\left(\sum_{k=1}^{n} |e_k^*(x)|^2\right)^{\frac{1}{2}} \leq C\|x\|.$$

Consider the family of operators $\mathcal{T}_n = \{e_k^* \otimes u : \|u\| = 1; k = 1, \ldots, n\}$. Let $x \in X$. Then for any $a_1, \ldots, a_n \in \mathbb{C}$ and every $u_1, \ldots, u_n \in X$ of norm one we have

$$\begin{aligned}
\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k a_k (e_k^* \otimes u_k)(x)\right\| &= \mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k a_k e_k^*(x) u_k\right\| \\
&\leq \sum_{k=1}^{n} \|a_k e_k^*(x) u_k\| \\
&= \sum_{k=1}^{n} |a_k||e_k^*(x)| \\
&\leq \left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |e_k^*(x)|^2\right)^{\frac{1}{2}} \\
&\leq C\left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}} \|x\|.
\end{aligned}$$

We conclude that $\mathcal{T}_n$ is semi-R-bounded with constant C and hence $\mathcal{T}_n$ is R-bounded for constant KC independent of n. This implies that X has type 2 as follows. Choose any $x_1, \dots, x_n \in X$ and write $x_k = \|x_k\| u_k$ where $\|u_k\| = 1$. Then

$$\begin{aligned}
\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k x_k\right\|^2\right)^{\frac{1}{2}} &= \left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k \|x_k\| u_k\right\|^2\right)^{\frac{1}{2}} \\
&= \left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k \|x_k\| e_k^*(e_k) u_k\right\|^2\right)^{\frac{1}{2}} \\
&= \left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k \|x_k\| (e_k^* \otimes u_k)(e_k)\right\|^2\right)^{\frac{1}{2}} \qquad (2.1) \\
&\leq KC\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k \|x_k\| e_k\right\|^2\right)^{\frac{1}{2}} \\
&\leq KC^2\left(\sum_{k=1}^{n}\|x_k\|^2\right)^{\frac{1}{2}}
\end{aligned}$$

Now, X has type 2 and cotype 2 and is therefore isomorphic to ℓ_2 by Kwapien's theorem [25].

Now suppose on the contrary that X has trivial type. We will show that X is a GT-space, i.e. any $T : X \longrightarrow \ell_2$ is 1-summing. Fix $T : X \longrightarrow \ell_2$ of norm one. Since X has cotype 2 we can equivalently show that any such T is 2-summing [11]. It suffices to check that for any $n \in \mathbb{N}$ and operator $S : \ell_2^n \longrightarrow X$ such that $\|S\| \leq 1$ we have $\pi_2(TS) \leq C$ where C does not depend on n [25]. One can assume that $TS : \ell_2^n \longrightarrow \ell_2^n$ and that TS is diagonal with respect to the canonical orthonormal basis (e_k) in ℓ_2^n, i.e. $TSe_k = \lambda_k e_k$ for some $\lambda_1, \dots, \lambda_n$. Then it suffices to show uniform boundedness of the Hilbert-Schmidt norms $\|TS\|_{HS} = \left(\sum_{k=1}^n \|TSe_k\|^2\right)^{\frac{1}{2}}$. Write $f_k^* = T^* e_k^* \in X^*$ and $f_k = Se_k \in X$. Consider $\{f_k^* \otimes u : k = 1, \dots, n; \|u\| = 1\}$. We will show that this family is semi-R-bounded with constant one. Take $u_1, \dots, u_n \in X$ of norm one and $a_1, \dots, a_n \in \mathbb{C}$. Then for $x \in X$ we have

$$\begin{aligned}
\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k a_k f_k^*(x) u_k\right\| &\leq \sum_{k=1}^{n}|a_k||f_k^*(x)| \\
&\leq \left(\sum_{k=1}^{n}|a_k|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}|e_k^*(Tx)|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{k=1}^{n}|a_k|^2\right)^{\frac{1}{2}}\|Tx\| \\
&\leq \left(\sum_{k=1}^{n}|a_k|^2\right)^{\frac{1}{2}}\|x\|
\end{aligned}$$

Therefore, $\{f_k^* \otimes u : k = 1, \dots, n; \|u\| = 1\}$ is R-bounded with constant K.

Since X has trivial type, it contains ℓ_1^n uniformly [19]. Hence, for fixed $M > 1$ and every $n \in \mathbb{N}$ there are $y_1, \ldots, y_n \in X$ with $\|y_k\| = 1$ for $1 \le k \le n$ such that

$$\sum_{k=1}^n |a_k| \le M \left\| \sum_{k=1}^n a_k y_k \right\| \tag{2.2}$$

Choose any scalars $b_1, \ldots, b_n$. Now using R-boundedness and Kahane's inequality for $p = 1$ with constant A we have

$$\begin{aligned}
\sum_{k=1}^n |b_k||\lambda_k| &= \sum_{k=1}^n |b_k||f_k^*(f_k)| \\
&\le M \left(\mathbb{E} \left\| \sum_{k=1}^n \epsilon_k b_k f_k^*(f_k) y_k \right\|^2 \right)^{\frac{1}{2}} \\
&= M \left(\mathbb{E} \left\| \sum_{k=1}^n \epsilon_k b_k (f_k^* \otimes y_k)(f_k) \right\|^2 \right)^{\frac{1}{2}} \\
&\le KM \left(\mathbb{E} \left\| \sum_{k=1}^n \epsilon_k b_k f_k \right\|^2 \right)^{\frac{1}{2}} \\
&\le KM \left(\mathbb{E} \left\| \sum_{k=1}^n \epsilon_k b_k e_k \right\|^2 \right)^{\frac{1}{2}} \\
&\le KM \left(\sum_{k=1}^n |b_k|^2 \right)^{\frac{1}{2}}
\end{aligned}$$

Thus,

$$\left(\sum_{k=1}^n |\lambda_k|^2 \right)^{\frac{1}{2}} \le KM$$

and so $\|TS\|_{HS} \le KM$. Therefore, any operator $T : X \longrightarrow \ell_2$ is 2-summing. This completes the proof of (i) implies (ii).

Now we will show that (ii) implies (i). Suppose that X is a GT-space of cotype 2, and that $\mathcal{T}$ is a family of semi-R-bounded operators. We will show that $\mathcal{T}$ is R-bounded. Since X is separable, there is a quotient map $Q : \ell_1 \longrightarrow X$. First, we show that any semi-R-bounded family of operators from ℓ_1 into X is already R-bounded. Let $\mathcal{S}$ be such a family with semi-R-boundedness constant one. Suppose $S_1, \ldots, S_n \in \mathcal{S}$ and $x_1, \ldots, x_n \in \ell_1$. Then $x_k = \sum_{j=1}^\infty \xi_{jk} e_j$ where (e_j) is the canonical basis of ℓ_1.

Let us denote by C the constant in the Kahane-Khintchine inequality for any Banach space:

$$\left(\mathbb{E} \left\| \sum_{k=1}^n \epsilon_k x_k \right\|^2 \right)^{\frac{1}{2}} \le C \mathbb{E} \left\| \sum_{k=1}^n \epsilon_k x_k \right\|.$$

Thus

$$\left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k S_k x_k\right\|^2\right)^{\frac{1}{2}} \leq C\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k S_k x_k\right\|$$

Then

$$\begin{aligned}\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k S_k x_k\right\| &= \mathbb{E}\left\|\sum_{k=1}^n \epsilon_k S_k \sum_{j=1}^\infty \xi_{jk} e_j\right\| \\ &\leq \sum_{j=1}^\infty \mathbb{E}\left\|\sum_{k=1}^n \epsilon_k \xi_{jk} S_k e_j\right\| \\ &\leq \sum_{j=1}^\infty \left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k \xi_{jk} S_k e_j\right\|^2\right)^{\frac{1}{2}} \\ &\leq \sum_{j=1}^\infty \left(\sum_{k=1}^n |\xi_{jk}|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Combining and using the Khintchine inequality again we obtain

$$\begin{aligned}\left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k S_k x_k\right\|^2\right)^{\frac{1}{2}} &\leq C^2 \sum_{j=1}^\infty \mathbb{E}\left|\sum_{k=1}^n \epsilon_k \xi_{jk}\right| \\ &= C^2\mathbb{E}\left(\sum_{j=1}^\infty \left|\sum_{k=1}^n \epsilon_k \xi_{jk}\right|\right) \\ &= C^2\mathbb{E}\left\|\sum_{j=1}^\infty \sum_{k=1}^n \epsilon_k \xi_{jk} e_j\right\|_{\ell_1} \\ &= C^2\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k \sum_{j=1}^\infty \xi_{jk} e_j\right\| \\ &= C^2\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k x_k\right\|\end{aligned}$$

Combining the previous two computations gives that $\mathcal{S}$ is R-bounded.

Now let $\mathcal{T}$ be a family of operators on X with semi-boundedness constant one. Let $Q : \ell_1 \to X$ be a quotient map and note that the family $\mathcal{S} = \{TQ : T \in \mathcal{T}\}$ is R-bounded with some constant B by the above calculation.

We will apply a characterization of GT-spaces of cotype 2 due to Pisier [19].

Proposition 2.3 (Pisier)**.** *X is a GT-space of cotype 2 if and only if there is a constant $C > 0$ such that for any $n \in \mathbb{N}, x_1, \ldots, x_n \in X$ there are $y_1, \ldots, y_n \in \ell_1$ such that $Qy_k = x_k, k = 1, \ldots, n$ and*

$$\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k y_k\right\| \leq C\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k x_k\right\| \tag{2.3}$$

Now take $n \in \mathbb{N}, T_1, \ldots, T_n \in \mathcal{T}$ and $x_1, \ldots, x_n \in X$. Choose $y_1, \ldots, y_n \in \ell_1$ according to 2.3. Then

$$\begin{aligned}\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k T_k x_k\right\| &= \mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k T_k Q y_k\right\| \\ &\leq B\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k y_k\right\| \\ &\leq CB\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k x_k\right\|\end{aligned}$$

Thus, $\mathcal{T}$ is R-bounded. The proof is complete. □

For a set $\mathcal{T}$ of bounded linear operators we will use the notation $\mathcal{T}^* = \{T^* : T \in \mathcal{T}\}$.

Lemma 2.4. (i) *If $\mathcal{T}$ is R-bounded then $\mathcal{T}^{**}$ is R-bounded (with the same constant).*

(ii) *If $\mathcal{T}$ is WR-bounded then $\mathcal{T}^*$ and $\mathcal{T}^{**}$ are WR-bounded (with the same constant).*

(iii) *If $\mathcal{T}$ is semi-R-bounded then $\mathcal{T}^{**}$ is semi-R-bounded (with the same constant).*

Proof. The proofs of (i) and (iii) are similar. For (i) suppose $T_1, \ldots, T_n \in \mathcal{T}$ and that $\mathcal{T}$ has R-boundedness constant one. Let $\Omega = \{-1, 1\}^n$ with $\mathbb{P}$ normalized counting measure on Ω. Let ϵ_k be the sequence of coordinate maps on Ω. Let Rad $(\Omega; X)$ be the subspace of $L_2(\Omega, \mathbb{P}; X)$ generated by the functions $\epsilon_k \otimes x$ for $1 \leq k \leq n$ and $x \in X$ (this space is isomorphic to X^n). Then Rad $(\Omega; X^{**})$ can be identified naturally with a subspace of Rad $(\Omega; X)^{**}$. Consider the map $\mathbf{T}$: Rad $(\Omega; X) \to$ Rad $(\Omega; X)$ defined by

$$\mathbf{T}\left(\sum_{k=1}^{n} \epsilon_k \otimes x_k\right) = \sum_{k=1}^{n} \epsilon_k \otimes T_k x_k.$$

Then $\|\mathbf{T}\| \leq 1$ and so $\|\mathbf{T}^{**}\| \leq 1$ and (i) follows.

Let us now prove (ii). Suppose $\mathcal{T}$ is WR-bounded with constant one and $T_1, \ldots, T_n \in \mathcal{T}$. Suppose $x_1^*, \ldots, x_n^* \in X^*$ are such that

$$\left(\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k x_k^*\right\|^2\right)^{\frac{1}{2}} \leq 1.$$

Then, using the identification of Rad (Ω, X^{**}) as the bidual of Rad (Ω, X) we observe that the set of functions of the form $\sum_{k=1}^{n} \epsilon_k x_k^{**}$ in Rad (Ω, X^{**}) such that

$$\sum_{k=1}^{n} |\langle T_k^* x_k^*, x_k^{**}\rangle| \leq 1$$

is weak*-closed and contains the unit ball of Rad (Ω, X). By Goldstine's theorem it contains the unit ball of Rad (Ω, X^{**}) and this implies that $\mathcal{T}^*$ is WR-bounded with constant one. □

Now it is time to give an example of a family of operators that is uniformly bounded but not WR-bounded. The previous lemma will imply that the corresponding dual family is semi-R-bounded but not WR-bounded.

Example. Let $X = \ell_p, 1 \leq p < 2$. Pick any non-zero element $x \in X$ and choose $u^* \in X^*$ of norm one such that $u^*(x) \neq 0$. Define $T_k = u^* \otimes e_k$ where (e_k) is the canonical basis of X. The family $\{T_k\}$ is uniformly bounded, $\|T_k\| = 1$, but we will show that it is not wR-bounded. Consider the dual basis (e_k^*) in $(\ell_p)^*$. Then

$$\sum_{k=1}^{n} |\langle T_k x, e_k^* \rangle| = \sum_{k=1}^{n} |\langle u^*(x) e_k, e_k^* \rangle| = n|u^*(x)| \tag{2.4}$$

On the other hand, we have

$$\left(\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k x\right\|^2\right)^{\frac{1}{2}} \left(\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k e_k^*\right\|^2\right)^{\frac{1}{2}} = \|x\| n^{\frac{1}{2}} n^{1/q} \tag{2.5}$$

Here q satisfies $1/p + 1/q = 1$. If $p < 2$ then $q > 2$ and $\frac{1}{2} + 1/q < 1$, so for $1 \leq p < 2$ the family $\{T_k\}$ can not be wR-bounded.
We have $T_k^* = e_k^{**} \otimes u^*$ on $X^* = \ell_q$ where $2 < q \leq \infty$. Consider $q \neq \infty$. Since by reflexivity $T_k^{**} = T_k$ and using 2.4 we see that $\{T_k^*\}$ is not WR-bounded. However, X^* has type 2 and hence $\{T_k^*\}$ is semi-R-bounded by 2.1.

3. The main results

Suppose X is any Banach space. We shall say that a sequence $\mathcal{T} = (T_k)_{k=1}^{\infty}$ is an *approximating sequence* if $\lim_{k\to\infty} \|x - T_k x\| = 0$ for every $x \in X$. We will say that $\mathcal{T}$ is compact (relatively, weakly compact) if each T_k is compact (relatively, weakly compact). We will say that $\mathcal{T}$ is commuting if we have $T_k T_l = T_l T_k$ for $l, k \in \mathbb{N}$.

If $\mathcal{T}$ is a commuting approximating sequence, let us define the subspace $E_{\mathcal{T}}$ of X^* to be the closed linear span of $\cup_k T_k^*(X^*)$. The following Lemma is trivial.

Lemma 3.1. *If $\mathcal{T}$ is a commuting approximating sequence then $E_{\mathcal{T}}$ is a norming subspace of X^*, i.e. for some C we have*

$$\|x\| \leq C \sup_{x^* \in B_{E_{\mathcal{T}}}} |x^*(x)| \qquad x \in X$$

and $(T_n^*|_{E_{\mathcal{T}}})_{n=1}^{\infty}$ is an approximating sequence for $E_{\mathcal{T}}$.

Let us recall that a Banach space X has property (V) of Pełczyński if every unconditionally converging operator $T : X \to Y$ is weakly compact. The spaces $C(K)$ have property (V) [17] and more generally any C^*-algebra has property (V) [18]. The disk algebra $A(\mathbb{D})$ also has property (V) [14], [10]; see also [23]. We also recall that a Banach space X is said to have property (V^*) if whenever (x_n) is a bounded sequence in X then either:

(i) (x_n) has a subsequence which is weakly Cauchy, or
(ii) (x_n) has a subsequence (y_n) such that for some sequence (y_n^*) in X^* and $\delta > 0$ we have $|y_n^*(y_n)| \geq \delta$ and

$$\left\|\sum_{k=1}^{n} a_k y_k^*\right\| \leq \max_{1\leq k\leq n} |a_k| \qquad a_1, \ldots, a_n \in \mathbb{C},\ n \in \mathbb{N}.$$

Property (V^*) was introduced by Pełczyński [17]. We note that Bombal [4] shows that every Banach lattice not containing c_0 has property (V^*). Any subspace of a space with property (V^*) also has property (V^*).

Lemma 3.2. *Let X, Y be Banach spaces and let $\mathcal{T} = (T_k)_{k=1}^\infty$ be any sequence of operators in $\mathcal{L}(X,Y)$. Suppose either (i) $\mathcal{T}$ is semi-R-bounded or (ii) $\mathcal{T}$ is WR-bounded and Y has property (V^*). Then for every $x \in X$ the sequence $(T_kx)_{k=1}^\infty$ has a weakly Cauchy subsequence.*

Proof. If not, by passing to a subsequence we can suppose $(T_kx)_{k=1}^\infty$ is equivalent to the canonical ℓ_1-basis ([21], [22]). If $\mathcal{T}$ is semi-R-bounded we observe that for some C we have

$$\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k a_k T_k x\right\| \le C\left(\sum_{k=1}^n |a_k|^2\right)^{\frac12}\|x\|, \qquad a_1,\dots,a_n \in \mathbb{C},\ n\in\mathbb{N}.$$

This gives a contradiction.

In case (ii), we can pass to a subsequence and assume the existence of $y_n^* \in Y^*$ such that

$$\left\|\sum_{k=1}^n a_k y_k^*\right\| \le \max_{1\le k\le n}|a_k| \qquad a_1,\dots,a_n\in\mathbb{C},\ n\in\mathbb{N}$$

and $|y_n^*(Tx_n)| \ge \delta > 0$ for all n. Then

$$\begin{aligned} n\delta &\le \sum_{k=1}^n |y_k^*(T_kx)| \\ &\le C\left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k x\right\|^2\right)^{\frac12}\left(\mathbb{E}\left\|\sum_{k=1}^n \epsilon_k y_k^*\right\|^2\right)^{\frac12} \\ &\le C\sqrt{n}. \end{aligned}$$

This also yields a contradiction. □

Theorem 3.3. *Let X be a Banach space with a commuting weakly compact approximating sequence $\mathcal{T}$. Suppose either that (i) $\mathcal{T}$ is semi-R-bounded and X is weakly sequentially complete or (ii) $\mathcal{T}$ is WR-bounded and X has property (V^*). Then X is isomorphic to a dual space.*

Proof. In either case we consider the family $\mathcal{T}^{**} \subset \mathcal{L}(X^{**}, X)$. By Lemma 3.2 for each $x^{**} \in X^{**}$ we can find a subsequence $T_{k_n}^{**}x^{**}$ so that $T_{k_n}^{**}(x^{**})$ is weakly convergent to some $y \in X$. Then for $x^* \in X^*$

$$x^*(T_ky) = \lim_{n\to\infty} x^*(T_kT_n^{**}x^{**}) = \lim_{n\to\infty} x^*(T_nT_k^{**}x^{**})$$

so that $T_ky = T_k^{**}x^{**}$. Hence $\lim_{k\to\infty}\|y - T_k^{**}x^{**}\| = 0$.

We now show that $E_{\mathcal{T}}^*$ can be identified with X. Clearly X canonically embeds in $E_{\mathcal{T}}^*$ since $E_{\mathcal{T}}$ is norming. If $f^* \in E_{\mathcal{T}}^*$ then by the Hahn-Banach theorem there exists $x^{**} \in X^{**}$ with $\|x^{**}\| = \|f^*\|$ and $x^{**}(x^*) = f^*(x^*)$ for $x^* \in E_{\mathcal{T}}$. Let $y = \lim_{k\to\infty} T_k^{**}x^{**}$. Then for $x^* \in E_{\mathcal{T}}$,

$$x^*(y) = \lim_{k\to\infty} x^*(T_k^{**}x^{**}) = \lim_{k\to\infty} x^{**}(T_k^*x^*) = f^*(x^*).$$

Hence $E_{\mathcal{T}}^* = X$. □

Theorem 3.4. *The space $L_1(0,1)$ does not have a commuting weakly compact approximating sequence which is either semi-R-bounded or WR-bounded.*

Proof. L_1 is not a dual space [25]. □

Of course a semi-R-bounded sequence in L_1 is actually R-bounded.

Theorem 3.5. *Let X be a separable Banach space with property (V). If X has a commuting weakly compact approximating sequence which is WR-bounded, then X^* is separable, and (T_n^*) is a commuting approximating sequence for X^*.*

Proof. Since X has (V), it follows that X^* has property (V^*). We show that $(T_n^*)_{n=1}^\infty$ is an approximating sequence for X^*. Indeed $T_n^*x^*$ converges weak* to x^* and it must have a weakly convergent subsequence by Lemma 3.2. Hence $x^* \in E_{\mathcal{T}}$ so $X^* = E_{\mathcal{T}}$. Now $T_n^*(B_{X^*})$ is weakly compact by Gantmacher's theorem also and weak*-metrizable, hence norm separable. Thus X^* is separable. □

Corollary 3.6. *If K is an uncountable compact metric space then $C(K)$ has no WR-bounded commuting weakly compact approximating sequence. The disk algebra has no WR-bounded weakly compact approximating sequence.*

We now consider $C(K)$ when K is countable. In this case $C(K)$ is homeomorphic to a space $C(\alpha) = C([1,\alpha])$ where α is a countable ordinal. There is a characterization of such $C(K)$ due to Bessaga and Pełczyński [3].

Theorem 3.7 (Bessaga-Pełczyński)**.** *If $\alpha < \beta, C(\omega^\alpha \cdot k)$ is isomorphic to $C(\omega^\beta \cdot n)$ if and only if $\beta < \alpha \cdot \omega$. Consequently, $C(\omega^{\omega^\gamma})$, $0 \le \gamma < \omega_1$, is a complete list of representatives of the isomorphism classes of $C(K)$ for K a countable compact metric space.*

The following lemma can be obtained as an applications of ℓ_1-indices ([1], [5], [6]). However, for convenience of the reader we will give a direct proof by construction.

Lemma 3.8. *Let α be a countable ordinal with $\alpha \ge \omega^\omega$. Then there exists $f \in C(\alpha)^{**}$ so that whenever $f_n \in C(\alpha)$ converges to $f \in C(\alpha)^{**}$ weak* then for any $m \in \mathbb{N}$ there exist $n_1, \dots, n_m \in \mathbb{N}$ such that*

$$\left\| \sum_{k=1}^m \epsilon_k f_{n_k} \right\| \ge \tfrac{1}{2} m \qquad \epsilon_k = \pm 1, \quad k = 1, 2, \dots, m.$$

Proof. In this case $C(\alpha)^{**}$ can be identified with $\ell_\infty(\alpha)$. It is easy to see that it suffices to consider the case $\alpha = \omega^\omega$.

Consider $f \in X^{**}$ defined by $f(\sum_{k=0}^N \omega^k l_k) = (-1)^{\sum_{k=0}^N l_k}$ and $f(\omega^\omega) = 1$. Writing K for the space $[1, \omega^\omega]$ let $K^{(p)}$ denote the $p-$th derived set of K. Then $K^{(p)}$ consists of all ordinals of the form $\sum_{k=p}^n \omega^k l_k$ together with ω^ω. For each $p \in \mathbb{N}$, $K^{(p)}$ is nonempty. Furthermore for each $\alpha \in K^{(p)}$ and every open neighborhood V of α we have that f takes both values ± 1 on $V \cap K^{(p-1)}$.

Let $f_n \in C(K)$ be any sequence such that (f_n) converges to f weak*.

Fix $0 < \delta < \frac{1}{2}$ and $m \in \mathbb{N}$. We construct $(f_{n_1}, \dots, f_{n_m})$ inductively. We start from $K^{(m)}$. By definition of f we can pick $\alpha_1^1, \alpha_2^1 \in K^{(m)}$ such that $f(\alpha_j^1) = (-1)^j$ for $j = 1, 2$. Then find $n_1 \in \mathbb{N}$ such that $|f_{n_1}(\alpha_j^1) - (-1)^j| < \delta$. Since f_{n_1} is continuous we can choose open neighborhoods U_j^1 of α_j^1 such that $|f_{n_1}(\alpha) - (-1)^j| < \delta$ for all $\alpha \in U_j^1$.

For the inductive step, suppose that $(n_j)_{j=1}^k$, $(\alpha_j^k)_{j=1}^{2^k}$ and open sets $(U_j^k)_{j=1}^{2^k}$ have been chosen so that $\alpha_j^k \in U_j^k$. Then for $i = 1, \dots, 2^k$ find points $\alpha_{2i-1}^{k+1}, \alpha_{2i}^{k+1} \in U_i^k \cap K^{(m-k+1)}$ with $f(\alpha_j^{k+1}) = (-1)^j$. By pointwise convergence, we can select $n_{k+1} > n_k$ such that $|f_{n_{k+1}}(\alpha_j^{k+1}) - (-1)^j| < \delta$. Since $f_{n_{k+1}}$ is continuous, there are neighborhoods $U_{2i-1}^{k+1}, U_{2i}^{k+1} \subset U_i^k, i = 1, \dots, 2^k$ such that for all $\alpha \in U_j^{k+1}$ we have $|f_{n_{k+1}}(\alpha) - (-1)^j| < \delta$.

In the m-th iteration this will give 2^m neighborhoods and m functions $f_{n_1}, \dots, f_{n_m}$ so that for any $\epsilon_1, \dots, \epsilon_m \in \{-1, +1\}$ there is an α contained in one of these neighborhoods such that $|f_k(\alpha) - \epsilon_k| < \delta$ for all $k = 1, \dots, m$. Hence

$$\| \sum_{k=1}^m \epsilon_k f_{n_k} \| \geq (1 - \delta) m.$$

□

Theorem 3.9. *Let K be a compact metric space. Suppose there is an R-bounded commuting weakly compact approximating sequence in $C(K)$. Then $C(K)$ is isomorphic to c_0.*

Proof. By Corollary 3.6 we need only consider the case when K is countable. By Theorem 3.7 it suffices to consider the case when $K = [1, \alpha]$ where $\alpha \geq \omega^\omega$. Pick $f \in C(K)^{**}$ satisfying the hypotheses of Lemma 3.8.

Suppose (T_n) is an R-bounded weakly compact approximating sequence for $C(K)$. Then (T_n^*) is an approximating sequence for $C(K)^*$ by Theorem 3.5 and hence $T_n^{**} f$ converges to f weak*. It follows that for any m we can choose $n_1, \dots, n_m$ so that

$$\left\| \sum_{k=1}^m \epsilon_k T_{n_k}^{**} f \right\| \geq \tfrac{1}{2} m \qquad \epsilon_k = \pm 1.$$

Hence

$$\left(\mathbb{E} \left\| \sum_{k=1}^m \epsilon_k T_{n_k}^{**} f \right\|^2 \right)^{\frac{1}{2}} \geq \tfrac{1}{2} m.$$

This contradicts the fact that T_n is R-bounded (or even semi-R-bounded). □

Remark. We can replace the assumption of R-boundedness by the assumption that (T_n) and (T_n^*) are both semi-R-bounded. By Theorem 2.2 this hypothesis would imply that (T_n^*) is actually R-bounded and hence that (T_n) is WR-bounded. We only used the fact that (T_n) is both semi-R-bounded and WR-bounded.

Let us conclude by stating our main result with respect to semigroups. (Actually our results are somewhat stronger than stated below.)

Theorem 3.10. *Let X be a separable Banach space with an R-bounded strongly continuous semigroup $(T_t)_{t>0}$ consisting of weakly compact operators. Then if*

1. *$X = L_1(\mu)$ for some measure μ then X is isomorphic to ℓ_1 (i.e. μ is purely atomic).*
2. *$X = C(K)$ then X is isomorphic to c_0.*

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Department of Mathematics, University of Missouri-Columbia, Columbia, MO 65211, USA

E-mail address: `mathgr26@math.missouri.edu; nigel@math.missouri.edu; tamara@math.missouri.edu`