A SUSPENSION FLOW OVER THE FULL SHIFT WITH TWO DISTINCT MEASURES OF MAXIMAL ENTROPY

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Abstract. We give an explicit construction of a suspension flow with continuous roof function over a full shift which has two distinct measures of maximal entropy. This is a special case of our results on measures of maximal entropy for suspension flows over the full shift presented in [4].

1. Introduction

In this note we explicitly construct a continuous roof function $\rho : \Sigma \mapsto (0, \infty)$, where Σ is the full shift on four symbols, so that the suspension flow has two measures of maximal entropy (MMEs). This contrasts with the case of a suspension flow with Hölder continuous roof function in which case the MME is unique [6].

While it will be no surprise to experts in this area that examples of suspension flows with multiple MME exist for roof functions beyond Hölder regularity, it is instructive to have concrete examples of this phenomenon. To the best of our knowledge, no such examples appear in the literature except in the case when the base is non-compact [3, 7].

We remark that our example confirms another expected phenomenon, discussed recently in [1], which is that orbit equivalence for flows does not preserve uniqueness of the MME. To see this, note that all suspension flows over the same base are orbit equivalent. In particular, the example here with two MMEs is orbit equivalent to a suspension flow with constant roof function which has a unique MME.

In our recent preprint [4], we proved the following more general result.

Theorem ([4]). *Let Σ be the full shift on a finite alphabet, and let $Y \subset \Sigma$ be a positive entropy subshift of finite type. There exists a continuous function $\rho : \Sigma \mapsto (0, \infty)$ so that the set of MME for the suspension flow on $\mathrm{Susp}(\Sigma, \rho)$ is exactly the set of lifts to $\mathrm{Susp}(\Sigma, \rho)$ of the MMEs for the subshift of finite type Y.*

The existence of the example presented here can be deduced as an immediate corollary of this result, and the associated construction is a special case of the arguments in [4]. The advantage of the argument presented in this note is that it is simpler than the one in [4], and we obtain a shorter and more transparent proof of the non-uniqueness phenomenon. We refer the reader to [4] for an extended bibliography of examples of non-uniqueness of MME and equilibrium states in the discrete time setting.

We recall some facts about suspension flows which can be found in the book by Parry and Pollicott [6]. Such flows consists of a shift space (Σ, σ) on the base

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along with a continuous roof function $\rho : \Sigma \mapsto (0,\infty)$ which determines the time the flow takes to return to this base. There is a canonical identification between the invariant probability measures for the suspension flow and the shift map (in the base of the flow). Moreover, we can apply Abramov's formula to compute the entropy of an invariant measure $\tilde{\mu}$ for the suspension flow in terms of the entropy of the corresponding shift invariant measure μ, i.e. $h_{\tilde{\mu}} = \frac{h_\mu}{\int \rho\, d\mu}$.

We consider the pressure functional $P : C(\Sigma,\mathbb{R}) \mapsto \mathbb{R}$ on the base shift space. The topological pressure satisfies the well-known variational principle, i.e. for any $g \in C(\Sigma,\mathbb{R})$ we have

$$P(g) = \sup_\mu \left\{ h_\mu + \int_\Sigma g\, d\mu \right\}, \tag{1.1}$$

where the sup is taken over all σ-invariant probability measures (see [8] for details). The measures which realize the supremum are called the equilibrium states for g.

Since $\rho > 0$, it follows from basic properties of topological pressure that there exists a unique constant c such that $P(-c\rho) = 0$. Now let μ be an equilibrium state for $-c\rho$. Then for any other σ-invariant measure ν we see that

$$0 = h_\mu + \int -c\rho d\mu \geq h_\nu + \int -c\rho d\nu,$$

where equality holds if and only if ν is also an equilibrium state for $-c\rho$. Thus,

$$\frac{h_\mu}{\int \rho d\mu} \geq \frac{h_\nu}{\int \rho d\nu},$$

and by Abramov's formula, we have $h_{\tilde{\mu}} \geq h_{\tilde{\nu}}$. Therefore, any MME for the suspension flow over (Σ,σ) with roof function ρ corresponds to an equilibrium state for $-c\rho$.

The preceding discussion reduces our problem to finding a continuous function $g : \Sigma \mapsto (-\infty, 0)$ such that $P(g) = 0$ and g has two equilibrium states, and setting the roof function as $\rho = -g$. To construct such a function g, we pick a subshift Y of Σ with two measures of maximal entropy and we define g in such a way that the MME for Y are the equilibrium states for g.

Our inspiration comes from the work of Hofbauer [2]. He provides an explicit example of a function on a full shift on two symbols with two equilibrium states. However, one of them must necessarily be a point mass measure, and hence his function is not bounded away from zero. We construct our function g on a full shift on four symbols, and we pick Y to be a subshift which is the union of two disjoint copies of the full shift on two symbols, each with its own MME. Aided by the 'symmetry' of the subshift Y, which is convenient for our estimates, we define a continuous negative function g for which these measures are equilibrium states.

2. The construction

We consider the (two-sided) shift space Σ on the alphabet $\mathcal{A} = \{0,1,2,3\}$, which is the set of all bi-infinite sequences $\xi = (\xi_n)_{n=-\infty}^{\infty}$ where $\xi_n \in \mathcal{A}$ for all $n \in \mathbb{Z}$. We endow Σ with the Tychonov product topology which makes Σ a compact metrizable space (see e.g. [5] for details). The shift map $\sigma : \Sigma \to \Sigma$ defined by $\sigma(\xi)_n = \xi_{n+1}$ is a continuous map on Σ. For a word $(\xi_0, ..., \xi_{n-1}) \in \mathcal{A}^n$ we denote by $[\xi_0 \cdots \xi_{n-1}] = \{\eta \in \Sigma : \eta_0 = \xi_0, ..., \eta_{n-1} = \xi_{n-1}\}$ the cylinder generated by $(\xi_0, ..., \xi_{n-1})$.

The purpose of this section is to construct a continuous function $g : \Sigma \mapsto \mathbb{R}$ with the following properties

(1) $g(\xi) \le -\log 2$ for all $\xi \in \Sigma$
(2) $P(g) = 0$
(3) g has more than one equilibrium state.

In view of the discussion in the previous section, the suspension flow over (Σ, σ) with roof function $\rho = -g$ will have more than one measure of maximal entropy.

We split the alphabet $\mathcal{A} = \{0,1,2,3\}$ into two subsets $\mathcal{A}_0 = \{0,1\}$ and $\mathcal{A}_1 = \{2,3\}$. For $k \ge 1$ we let

$$M_k = \{\xi \in \Sigma : \exists i \in \{0,1\} \,\text{such that}\, \xi_0, ..., \xi_{k-1} \in \mathcal{A}_i \,\text{and}\, \xi_k \in \mathcal{A}_{1-i}\} \tag{2.1}$$

and $M_0 = \{\xi \in \Sigma : \exists i \in \{0,1\} \,\text{such that}\, \xi_j \in \mathcal{A}_i \,\text{for all}\, j\}$. Note that M_0 is a subshift of finite type with the transition matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Therefore, M_0 supports two measures of maximal entropy μ_0 and μ_1 with $h_{\mu_0} = h_{\mu_1} = \log 2$.

Consider a sequence of real numbers $(a_k)_{k=1}^{\infty}$ such that

- $a_k < -\log 2$ for all $k \ge 1$,
- $\lim\limits_{k\to\infty} a_k = -\log 2$,
- $\sum\limits_{k=1}^{\infty} 2^k e^{a_1+\ldots+a_k} \le 1$.

An example of such a sequence is given at the end of our argument. We define $g(\xi) = a_k$ if $\xi \in M_k$ and $g(\xi) = -\log 2$ if $\xi \in M_0$. Then $g : \Sigma \to \mathbb{R}$ is continuous since $\lim\limits_{k\to\infty} a_k = -\log 2$. Moreover, by the Variational Principle

$$P(g) \ge h_{\mu_0} + \int g \, d\mu_0 = \log 2 - \log 2 = 0. \tag{2.2}$$

It remains to show that $P(g) \le 0$. By definition, $P(g) = \lim\limits_{k\to\infty} \frac{1}{k} \log Z_k(g)$, where

$$Z_k(g) = \sum_{\xi_0,\ldots,\xi_{k-1}\in\mathcal{A}} \exp\sup\{S_k g(\eta) : \eta \in [\xi_0 \cdots \xi_{k-1}]\}, \tag{2.3}$$

using the notation

$$S_k g(\eta) = \sum_{j=0}^{k-1} g(\sigma^j \eta).$$

Note that g is defined 'symmetrically' with respect to the two transitive components of M_0 in the sense that on each M_k, there is a bijection between the points with $\xi_k \in \mathcal{A}_0$ and the points with $\xi_k \in \mathcal{A}_1$. Thus, we have

$$\begin{aligned} Z_k(g) &= \sum_{i=0}^{1} \sum_{\substack{\xi_{k-1}\in\mathcal{A}_i \\ \xi_0,\ldots,\xi_{k-2}\in\mathcal{A}}} \exp\sup\{S_k g(\eta) : \eta \in [\xi_0 \cdots \xi_{k-1}]\} \\ &= 2 \sum_{\substack{\xi_{k-1}\in\mathcal{A}_0 \\ \xi_0\cdots\xi_{k-2}\in\mathcal{A}}} \exp\sup\{S_k g(\eta) : \eta \in [\xi_0 \cdots \xi_{k-1}]\} \end{aligned}$$

For each cylinder $[\xi_0\cdots\xi_{k-1}]$ either all ξ_j are in $\mathcal{A}_0$ or there exists $0 \le r \le k-2$ such that $\xi_r \in \mathcal{A}_1$ and $\xi_{r+1},\dots,\xi_{k-1} \in \mathcal{A}_0$. We obtain

$$Z_k(g) = 2\sum_{\xi_0,\dots,\xi_{k-1}\in\mathcal{A}_0} \exp\sup\{S_k g(\eta) : \eta \in [\xi_0\cdots\xi_{k-1}]\} + 2\sum_{r=0}^{k-2} \sum_{\substack{\xi_j\in\mathcal{A},\, j<r \\ \xi_r\in\mathcal{A}_1 \\ \xi_{r+1}\cdots\xi_{k-1}\in\mathcal{A}_0}} \exp\sup\{\mathcal{A}_k g(\eta) : \eta \in [\xi_0\cdots\xi_{k-1}]\} \tag{2.4}$$

If all the initial terms $\xi_0,\dots,\xi_{k-1}$ are in $\mathcal{A}_0$, then $\sup\{S_k g(\eta) : \eta \in [\xi_0\cdots\xi_{k-1}]\}$ will be attained when $\eta_k, \eta_{k+1},\dots$ are also in $\mathcal{A}_0$. Since in this case $S_k g(\eta) = -k\log 2$, the first sum in (2.4) reduces to

$$\sum_{\xi_0,\dots,\xi_{k-1}\in\mathcal{A}_0} \exp(-k\log 2) = 2^k \exp(-k\log 2) = 1.$$

We turn our attention to the second sum. Again, 'symmetry' of g allows us to assume $\xi_r = 2$ and double the sum. We fix $r > 0$ and $\xi_0,\dots,\xi_{r-1} \in \mathcal{A}$ and consider

$$\sum_{\xi_{r+1},\dots,\xi_{k-1}\in\mathcal{A}_0} \exp\sup\{S_k g(\eta) : \eta \in [\xi_0\cdots\xi_{r-1}2\xi_{r+1}\cdots\xi_{k-1}]\}.$$

For $\eta \in [\xi_0\cdots\xi_{r-1}2\xi_{r+1}\cdots\xi_{k-1}]$,

$$\begin{aligned} S_k g(\eta) &= S_k g(\xi_0\cdots\xi_{r-1}2\xi_{r+1}\cdots\xi_{k-1}\eta_k\eta_{k+1}\cdots) \\ &= S_{r+1} g(\xi_0\cdots\xi_{r-1}2\xi_{r+1}\cdots\xi_{k-1}\eta_k\eta_{k+1}\cdots) + S_{k-r-1} g(\xi_{r+1}\cdots\xi_{k-1}\eta_k\eta_{k+1}\cdots) \end{aligned}$$

Notice that the value of $S_{r+1} g(\xi_0\cdots\xi_{r-1}2\xi_{r+1}\cdots\xi_{k-1}\eta_k\eta_{k+1}\cdots)$ does not depend on the values of $\xi_{r+1},\dots,\xi_{k-1} \in \mathcal{A}_0$ and $\eta_k,\eta_{k+1},\dots \in \mathcal{A}$. On the other hand, the largest possible value of $S_{k-r-1} g(\xi_{r+1}\cdots\xi_{k-1}\eta_k\eta_{k+1}\cdots)$ is attained when $\eta_k,\eta_{k+1},\dots \in \mathcal{A}_0$. In this case, $S_{k-r-1} g(\xi_{r+1}\cdots\xi_{k-1}\eta_k\eta_{k+1}\cdots) = (k-r-1)(-\log 2)$. Therefore, we may pick all the coordinates of η starting with k to be zero and obtain

$$\sup\{S_k g(\eta) : \eta \in [\xi_0\cdots\xi_{r-1}2\xi_{r+1}\cdots\xi_{k-1}]\} = S_{r+1} g(\xi_0\cdots\xi_{r-1}200\cdots) - (k-r-1)\log 2.$$

Similarly, when $r = 0$ we have

$$\sup\{S_k g(\eta) : \eta \in [2\xi_{r+1}\cdots\xi_{k-1}]\} = a_1 - (k-1)\log 2.$$

Therefore,

$$\begin{aligned} Z_k(g) &= 2 + 4\sum_{\xi_1,\dots,\xi_{k-1}\in\mathcal{A}_0} \exp[a_1 - (k-1)\log 2] \\ &\quad + 4\sum_{r=1}^{k-2} \sum_{\substack{\xi_{r+1},\dots,\xi_{k-1}\in\mathcal{A}_0 \\ \xi_0,\dots,\xi_{r-1}\in\mathcal{A}}} \exp[S_{r+1} g(\xi_0\cdots\xi_{r-1}200\cdots) - (k-r-1)\log 2] \\ &= 2 + 4e^{a_1} + 4\sum_{r=1}^{k-2} \sum_{\xi_0,\dots,\xi_{r-1}\in\mathcal{A}} \exp S_{r+1} g(\xi_0\cdots\xi_{r-1}200\cdots). \end{aligned} \tag{2.5}$$

To estimate the last sum we introduce some notation. Let

$$A_j = \sum_{\xi_1,\dots,\xi_{j-1}\in\mathcal{A}} \exp S_{j+1} g(0\xi_1\cdots\xi_{j-1}200\cdots) \quad \text{for } j \geq 2 \text{ and } A_1 = \exp[g(0200\cdots) + g(200\cdots)];$$

$$B_j = \sum_{\xi_1,\dots,\xi_{j-1}\in\mathcal{A}} \exp S_{j+1} g(2\xi_1\cdots\xi_{j-1}200\cdots) \quad \text{for } j \geq 2 \text{ and } B_1 = \exp[g(2200\cdots) + g(200\cdots)];$$

$$C_j = \sum_{\xi_0,\xi_1,\dots,\xi_{j-1}\in\mathcal{A}} \exp S_{j+1} g(\xi_0\xi_1\cdots\xi_{j-1}200\cdots) \quad \text{for } j \geq 1.$$

Clearly, $C_j = 2(A_j + B_j)$ for any $j \geq 1$. First, we show that for $j \geq 2$ we have

$$A_j = 2^{j-1} e^{a_1} e^{a_1+\dots+a_j} + \sum_{i=1}^{j-1} 2^{j-i} e^{a_1+\dots+a_{j-i}} B_i. \tag{2.6}$$

$$\begin{aligned}
A_j &= \sum_{\xi_1,\dots,\xi_{j-1}\in\mathcal{A}} \exp S_{j+1} g(0\xi_1\cdots\xi_{j-1}200\cdots) \\
&= 2\sum_{\xi_2,\dots,\xi_{j-1}\in\mathcal{A}} \exp S_{j+1} g(02\xi_2\cdots\xi_{j-1}200\cdots) + 2\sum_{\xi_2,\dots,\xi_{j-1}\in\mathcal{A}} \exp S_{j+1} g(00\xi_2\cdots\xi_{j-1}200\cdots) \\
&= 2e^{a_1}\sum_{\xi_2,\dots,\xi_{j-1}\in\mathcal{A}} \exp S_j g(2\xi_2\cdots\xi_{j-1}200\cdots) + 2\sum_{\xi_2,\dots,\xi_{j-1}\in\mathcal{A}} \exp S_{j+1} g(00\xi_2\cdots\xi_{j-1}200\cdots) \\
&= 2e^{a_1} B_{j-1} + 2\sum_{\xi_2,\dots,\xi_{j-1}\in\mathcal{A}} \exp S_{j+1} g(00\xi_2\cdots\xi_{j-1}200\cdots) \\
&= 2e^{a_1} B_{j-1} + 2^2\sum_{\xi_3,\dots,\xi_{j-1}\in\mathcal{A}} \exp S_{j+1} g(002\xi_3\cdots\xi_{j-1}200\cdots) \\
&\qquad + 2^2\sum_{\xi_3,\dots,\xi_{j-1}\in\mathcal{A}} \exp S_{j+1} g(000\xi_3\cdots\xi_{j-1}200\cdots) \\
&= 2e^{a_1} B_{j-1} + 2^2 e^{a_2+a_1}\sum_{\xi_3,\dots,\xi_{j-1}\in\mathcal{A}} \exp S_{j-1} g(2\xi_3\cdots\xi_{j-1}200\cdots) \\
&\qquad + 2^2\sum_{\xi_3,\dots,\xi_{j-1}\in\mathcal{A}} \exp S_{j+1} g(000\xi_3\cdots\xi_{j-1}200\cdots) \\
&= 2e^{a_1} B_{j-1} + 2^2 e^{a_2+a_1} B_{j-2} + 2^2\sum_{\xi_3,\dots,\xi_{j-1}\in\mathcal{A}} \exp S_{j+1} g(000\xi_3\cdots\xi_j 200\cdots) \\
&= 2e^{a_1} B_{j-1} + 2^2 e^{a_2+a_1} B_{j-2} + 2^3 e^{a_3+a_2+a_1} B_{j-3} + 2^3\sum_{\xi_4,\dots,\xi_{j-1}\in\mathcal{A}} \exp S_{j+1} g(0000\xi_4\cdots\xi_j 200\cdots) \\
&\vdots \\
&= 2e^{a_1} B_{j-1} + 2^2 e^{a_2+a_1} B_{j-2} + \dots + 2^{j-1} e^{a_{j-1}+\dots+a_1} B_1 + 2^{j-1}\exp S_{j+1} g(00\cdots 0200\cdots) \\
&= 2e^{a_1} B_{j-1} + 2^2 e^{a_2+a_1} B_{j-2} + \dots + 2^{j-1} e^{a_{j-1}+\dots+a_1} B_1 + 2^{j-1} e^{a_j+..+a_1+a_1} \\
&= \sum_{i=1}^{j-1} 2^{j-i} e^{a_1+\dots+a_{j-i}} B_i + 2^{j-1} e^{a_1} e^{a_1+\dots+a_j}
\end{aligned}$$

Similarly, one can show that

$$B_j = 2^{j-1} e^{a_{j+1}} e^{a_1+\dots+a_j} + \sum_{i=1}^{j-1} 2^{j-i} e^{a_1+\dots+a_{j-i}} A_i. \tag{2.7}$$

Combining (2.6) and (2.7) we obtain that for any $j \geq 2$,

$$C_j = 2^j (e^{a_1} + e^{a_{j+1}}) e^{a_1+\dots+a_j} + \sum_{i=1}^{j-1} 2^{j-i} e^{a_1+\dots+a_{j-i}} C_i. \tag{2.8}$$

We can also compute directly that

$$\begin{aligned} C_1 &= \sum_{\xi_1 \in \mathcal{A}} \exp S_2 g(\xi_0 200\cdots) \\ &= 2\exp S_2 g(0200\cdots) + 2\exp S_2 g(2200\cdots) \\ &= 2e^{2a_1} + 2e^{a_1+a_2} \\ &= 2e^{a_1}(e^{a_1} + e^{a_2}) < 1. \end{aligned}$$

Assume $C_i < 1$ for all $i \le j-1$. Since $a_1, a_{j+1} < -\log 2$ and $\sum_{k=1}^{\infty} 2^k e^{a_1+\ldots+a_k} \le 1$ we obtain that C_j must also be less than one. Hence, by induction $C_j \le 1$ for all j. Coming back to equation (2.5), for $k \ge 2$

$$Z_k(g) = 2 + 4e^{a_1} + 4\sum_{j=1}^{k-2} C_j < 2 + 2 + 4(k-2) = 4k - 4.$$

Therefore, $P(g) = \lim_{k\to\infty} \frac{1}{k}\log Z_k(g) \le \lim_{k\to\infty} \frac{1}{k}\log(4k-4) \le 0$.

Finally, we give an example of a sequence (a_k) which satisfies the required properties. We may take $a_i = -(1 + \frac{c}{\sqrt{i}})\log 2$, where $c \ge 2$. Then $a_i < -\log 2$ for all $i \ge 1$ and $\lim_{i\to\infty} a_i = -\log 2$. Also,

$$a_1 + \ldots + a_k = -\left(k + c\sum_{i=1}^{k} \frac{1}{\sqrt{i}}\right)\log 2 < -\left(k + c\sum_{i=1}^{k} \frac{1}{\sqrt{k}}\right)\log 2 \le -(k + c\sqrt{k})\log 2.$$

Using the standard integral estimate, one can show that whenever $c \ge 2$

$$\sum_{k=1}^{\infty} 2^k e^{a_1+\ldots+a_k} < \sum_{k=1}^{\infty} 2^{-c\sqrt{2}} \le 1.$$

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