Asymptotic behavior of the pressure function for Hölder potentials

Tamara Kucherenko, CCNY
(joint work with Anthony Quas)

November 5, 2022

Topological Pressure

"The main object of the thermodynamic formalism is to study the differentiability and analyticity properties of the function P [topological pressure], and the structure of the equilibrium states and Gibbs states."

- David Ruelle

Let $\phi : X \to \mathbb{R}$ be a continuous potential associated with a dynamical system (X, T).

The topological pressure of ϕ is defined by

$$P_{\text{top}}(\phi) = \sup_{\mu \in \mathcal{M}} \{\mathsf{h}_\mu + \int \phi \, d\mu\},$$

where $\mathcal{M}$ is the set of all T-invariant probability measures on X and h_μ is the measure-theoretic entropy of μ.

From the statistical physics point of view, $P_{\text{top}}(\phi)$ corresponds to the minimum of the free energy $E_\mu = -(\mathsf{h}_\mu + \int \phi \, d\mu)$.

A measure $\mu \in \mathcal{M}$ which minimizes the free energy (i.e. $P_{\text{top}}(\phi) = \mathsf{h}_\mu + \int \phi \, d\mu$) is called an equilibrium state of ϕ.

The Pressure Function

We study the pressure function of ϕ, $p_\phi(t) = P_{\text{top}}(t\phi) = \sup\{h_\mu + t\int \phi \, d\mu\}$ where t is a real valued parameter.

Statistical physics: $p_\phi(t)$ is a tool to observe an evolution of a system depending on a continuous external factor.
One interpretation of t is the inverse temperature of the system.

General Properties:

- Variational principle implies that $p_\phi(t)$ is convex.
- Monotonicity of $\psi \mapsto P_{\text{top}}(\psi)$ and the equality $P_{\text{top}}(\psi + c) = P_{\text{top}}(\psi) + c$ imply that $p_\phi(t)$ is Lipschitz.
- When $t \to \infty$ the term $\int \phi \, d\mu$ predominates so $p_\phi(t)$ has a slant asymptote.

There is nothing more! (for continuous potentials on compact systems)

The Case of Hölder Potentials

Hölder potentials are important since they correspond to "exponentially decaying interactions" in statistical physics.

Ruelle(1968)

If X is a mixing subshift of finite type then the pressure functional P_{top} acts real analytically on the space of Hölder potentials. Moreover, for each such potential there is only one equilibrium state.

This result coupled with Markov partitions and symbolic coding of Sinai allowed Bowen to describe the equilibrium states of Anosov diffeomorphisms.

When ϕ is Hölder $p_\phi(t)$ is analytic and strictly convex.

- Analyticity is the strongest possible regularity condition for real-valued functions.
- We enhance the strict convexity property.
- We completely characterize the asymptotic behavior of $p_\phi(t)$.

Asymptotic Behavior

(X, T) is a mixing subshift of finite type, $\phi : X \to \mathbb{R}$ is a Hölder potential.
We interpret t as the inverse temperature of the system.

What is the behavior of $p_\phi(t)$ as $t \to \infty$ (the temperature is lowered to zero)?

For each t the potential $t\phi$ has a unique equilibrium state μ_t.

The weak*-accumulation points of $\{\mu_t\}$ as $t \to \infty$ are called the ground states of ϕ.
Intuitively, the ground states should be supported on configurations of low complexity.

Contreras (2016): *For a generic Hölder potential on a mixing subshift of finite type the ground state is unique and supported on a periodic orbit.*

(built on an earlier work of **Morris (2008):** *generically ground states have zero entropy*)

It is easy to produce a Hölder potential whose ground state has positive entropy.
What about uniqueness?

Asymptotic Behavior

Brémont (2008): *any locally constant potential on a mixing subshift of finite type has a unique ground state (zero temperature measure).*

Chazottes, Hochman (2010): *there is a Hölder potential on a full shift with two distinct ground states.*

Although there might be multiple ground states, they all have the same free energy!

If μ_∞ is a ground state for ϕ then

$\int \phi \, d\mu_\infty = \max \left\{ \int \phi \, d\mu : \mu \in \mathcal{M} \right\}$ and $h_{\mu_\infty} = \max \left\{ h_\mu : \mu \text{ maximizes } \int \phi \, d\mu \right\}$

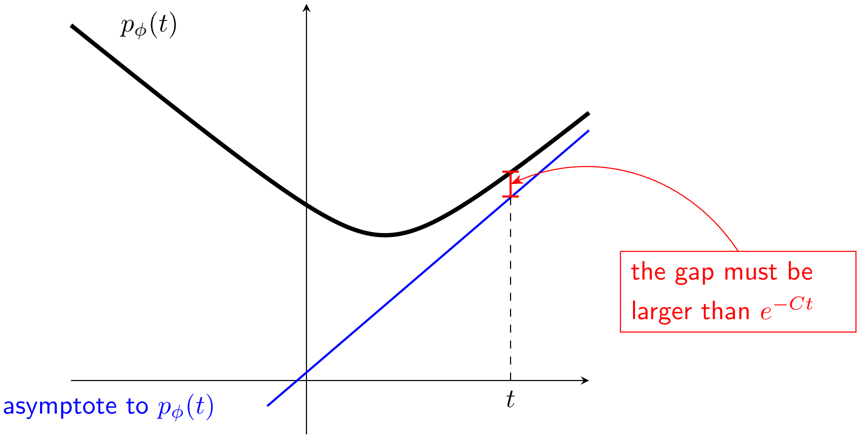
$p_\phi(t) = h_{\mu_t} + t \int \phi \, d\mu_t \approx h_{\mu_\infty} + t \int \phi \, d\mu_\infty$, the asymptote is $\ell(t) = h_{\mu_\infty} + t \int \phi \, d\mu_\infty$

free energy of μ_t — free energy of μ_∞

How fast the energy of the system can approach the energy of its ground state?

(How fast the pressure function can decrease towards its asymptote?)

Asymptotic Behavior



Asymptotic Behavior

Theorem 1

Let X be a mixing subshift of finite type with positive entropy. Let ϕ be a Hölder potential that is not cohomologous to a constant. Then there exist C and t_0 such that

$$p_\phi(t) \geq \ell(t) + e^{-Ct} \text{ for all } t \geq t_0,$$

where $\ell(t)$ is the asymptote of $p_\phi(t)$.

The exponential lower bound is the best one can hope for.

Example: Let $X = \{0,1\}^{\mathbb{Z}}$ and define $\phi(x) = \begin{cases} 0, & \text{if } x_0 = 0 \\ 1, & \text{if } x_0 = 1. \end{cases}$.

Then $p_\phi(t) = \log(1 + e^t)$, $\ell(t) = t$, and $e^{-t}(1 - e^{-t}) < p_\phi(t) - \ell(t) < e^{-t} \quad (\approx e^{-t})$.

Asymptotic Behavior

If ϕ is locally constant then the "gap" between $p_\phi(t)$ and $\ell(t)$ is exactly exponential.
If ϕ is Hölder we have an exponential lower bound on the "gap".

Can we get an exponential upper bound? **No.**

There is no upper bound!

Theorem 2

Suppose X is any mixing subshift of finite type with positive entropy and $f : \mathbb{R} \to \mathbb{R}$ is any convex function with asymptote $at + b$, where $0 \leq b < h_{\text{top}}(X)$. Then there exists a Hölder potential $\phi : X \to \mathbb{R}$ such that

- $p_\phi(t)$ *has asymptote* $at + b$;
- $p_\phi(t) > f(t)$ *for all sufficiently large* t.

Convexity

For a continuous potential the pressure function might be analytic and strictly convex, but uniqueness of equilibrium states fails.

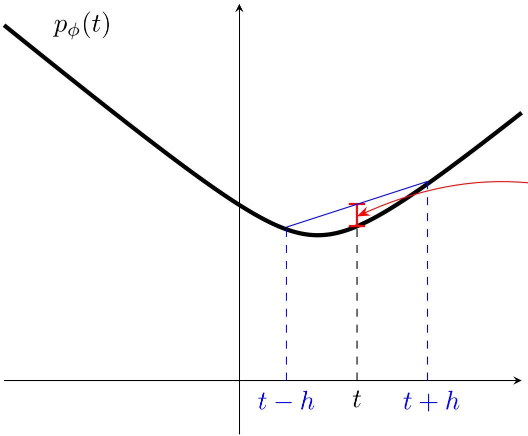
Leplaideur (2015): *an example of a continuous ϕ on a mixing subshift of finite type such that $p_\phi(t)$ is analytic, but there are two values of t for which $t\phi$ has two equilibrium states.*

K., Quas (2022): *for any upper semi-continuous function $N\colon \mathbb{R} \to \{1, \ldots, k, \infty\}$, there is a continuous ϕ on a full shift such that for $t > \alpha > 0$ $p_\phi(t)$ is analytic strictly convex and $t\phi$ has exactly $N(t)$ ergodic equilibrium states.*

For Hölder ϕ we have uniqueness of equilibrium states and $p_\phi(t)$ is strictly convex.

Strictly convex real-analytic functions could be "almost flat" on some intervals.
We show that this is not possible for the pressure function of a Hölder potential.

Convexity



the gap must be larger than $c_1 e^{-c_2|t|/h}$

Convexity

Theorem 3

Let X be a mixing subshift of finite type with positive entropy. Let ϕ be a Hölder potential that is not cohomologous to a constant. Then there exist c_1 and c_2 such that for any $t \in \mathbb{R}$, and any $h \in \mathbb{R}^+$,

$$\frac{p_\phi(t+h) + p_\phi(t-h)}{2} - p_\phi(t) > c_1 e^{-c_2|t|/h}.$$

We interpret this as a quantitative lower bound on the curvature, where the curvature bounds improve (a lot) when one considers coarse intervals.

Motivation

The pressure function is used to obtain information about

- Lyapunov exponents
- dimension,
- multifractal spectra,
- natural invariant measures...

Any new general properties could have far reaching consequences.

This was not the motivation for this work.

Katok's Flexibility Program

"under properly understood general restrictions, within a fixed class of smooth dynamical systems dynamical invariants take arbitrary values." - Anatole Katok

Goals of the flexibility program:

(1) understand the most general constraints which define a common class of systems;

(2) build tools to freely manipulate the dynamical data within those constraints.

- Bochi, Katok, Rodriguez Hertz 2019: *Flexibility of Lyapunov exponents.*
- Erchenko 2019: *Flexibility of Lyapunov exponents with respect to two classes of measures on the torus.*
- Alsedà, Misiurewicz, Pérez 2020: *Flexibility of entropies for piecewise expanding unimodal maps*
- Abrams, Katok, Ugarcovici 2020: *Flexibility of measure-theoretic entropy of boundary maps associated to Fuchsian groups.*
- Carrasco,Saghin 2021:*Extended flexibility of Lyapunov exponents for Anosov maps*

Flexibility of the Pressure Function

We obtain the flexibility of the pressure function for compact symbolic systems.

K., Quas 2021: *Let* $\alpha > 0$ *and let* $f(t)$ *be a convex Lipschitz function on* (α, ∞) *which has a slant asymptote* $at + b$ *with* $b \geq 0$. *Then there exists a full shift on a finite alphabet and a continuous potential* ϕ *such that* $p_\phi(t) = f(t)$ *for all* $t \in (\alpha, \infty)$.

What happens to Hölder potentials?

If we replace "Lipschitz" by "analytic", and "convex" by "strictly convex", can we make ϕ Hölder? **No**.

Flexibility Question

We uncovered two new properties of the pressure function for Hölder potentials: **asymptotic behavior** and **convexity estimates**.

There could be many more...

Question

Is it possible to give a nice characterization of the functions which arise as pressure functions for Hölder potentials on mixing subshifts of finite type?

Thank you!