Thermodynamic Formalism for Coded Shifts

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Symbolic Models

Idea: Use a well understood symbolic system to study a not so well understood system.

Uniformly Hyperbolic Systems can be modeled by subshifts of finite type, while non-uniformly hyperbolic systems can be modeled by countable Markov shifts.

• For an Axiom A diffeo the hyperbolic set is modeled by a subshift of finite type. [Bowen,1975]

Subshifts of finite type are the most understood dynamical systems.

• For a $C^{1+\beta}$ diffeo with $h_{\text{top}} > 0$ the set which is of full measure for all χ-hyperbolic measures is modeled by a countable Markov shift. [Sarig 2013]

Due to Sarig, Mauldin-Urbanski we have a good understanding of countable shifts.

Can non-uniformly Hyperbolic Systems be modeled by coded shifts?

Advantage: coded shifts are compact.

• For a skew-products with concave interval fiber maps over a shift the maximal invariant set where the projections of orbits stay in a given region is modeled by a coded shift. [Diaz, Gelfert, Rams, 2022]

I will discuss recent progress in understanding coded shifts.

Compact Shift Spaces

Let $\mathcal{A}$ be a finite alphabet and consider the product space $\mathcal{A}^{\mathbb{Z}} = \{(x_i)_{i\in\mathbb{Z}} : x_i \in \mathcal{A}\}$.

This is a compact space in the product topology and a compatible metric is given by

$$d(x,y) = 2^{-\min\{|i| : x_i \neq y_i\}}.$$

The shift map $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined as $(\sigma x)_i = x_{i+1}$ is a homeomorphism.

A subshift X on an alphabet $\mathcal{A}$ is a closed shift-invariant subset of $\mathcal{A}^{\mathbb{Z}}$.

Denote by $\mathcal{L}_n(X)$ the set of words of length n which appear in X, then $\mathcal{L}(X) = \bigcup \mathcal{L}_n(X)$ is the language of X.

The topological entropy of X is

$$h_{\text{top}}(X) = \lim_{n\to\infty} \frac{\log |\mathcal{L}_n(X)|}{n}$$

Compact Shift Spaces

Constructing a subshift using forbidden sets:

Let $\mathcal{A}^* = \bigcup \mathcal{A}^n$ and $\mathcal{F} \subset \mathcal{A}^*$. $X_\mathcal{F} = \{x \in \mathcal{A}^\mathbb{Z} : x_i \dots x_j \notin \mathcal{F} \text{ for } i \leq j\}$ is a subshift.

X is a subshift of finite type (SFT) if $X = X_\mathcal{F}$ for some finite $\mathcal{F}$.

Examples (on $\mathcal{A} = \{0, 1\}$):

1. $\mathcal{F} = \{11\}$, then $X_\mathcal{F}$ is the golden mean shift (SFT), $h_{\text{top}}(X) = \log \frac{1+\sqrt{5}}{2}$.
2. $\mathcal{F} = \{10^k 1 : k \text{ is odd}\}$, then $X_\mathcal{F}$ is the even shift (not an SFT).

Constructing a subshift using generating sets:

Let $\mathcal{G} \subset \mathcal{A}^*$. $X(\mathcal{G}) = \overline{\{\cdots g_{-2} g_{-1} g_0 g_1 g_2 g_3 \dots : g_i \in \mathcal{G}\}}$ is a subshift

Examples (on $\mathcal{A} = \{0, 1\}$):

1. $\mathcal{G} = \{0, 01\}$, then $X(\mathcal{G})$ is the golden mean shift.
2. $\mathcal{G} = \{1, 00\}$, then $X(\mathcal{G})$ is the even shift.
3. $\mathcal{G} = \{0^k 1 : k \text{ is even}\}$, then $X(\mathcal{G})$ is the even shift.

Equivalent Definitions of Coded Shifts

Definition 1 [Blanchard and Hansel in 1986]

A shift space X is coded if there exists a $\mathcal{G} \subset \mathcal{L}(X)$ (called a generating set) such that $X = X(\mathcal{G})$ is the smallest shift space that contains all bi-infinite concatenations of the elements of $\mathcal{G}$.

Examples: transitive SFTs and sofic shifts (factors of SFTs), S-gap shifts, β-shifts, Dyck shift

Definition 2

Every shift space has a representation as a countable directed labeled graph. Coded shifts are precisely those shift spaces which have a strongly connected representation.

Sofic shifts are represented by a finite directed labeled graph. If $\mathcal{G}$ is finite then $X(G)$ is sofic.

Definition 3 [Krieger, 2000]

X is coded if it can be approximated by an increasing sequence of irreducible SFTs.

Generating Sets

The same coded shift can be generated by many generating sets. Some generating sets are "better" than others!

Fix a generating set $\mathcal{G} \subset \mathcal{A}^*$. Let $X = X(\mathcal{G})$ be the coded shift generated by $\mathcal{G}$. The concatenation set of X is the set of all bi-infinite concatenations of elements of $\mathcal{G}$:

$$X_{\text{con}}(\mathcal{G}) = \{x \in \mathcal{A}^{\mathbb{Z}} : \exists (k_i)_{i\in\mathbb{Z}} \subset \mathbb{Z}, k_i < k_{i+1} \text{ and } x_{k_i} \ldots x_{k_{i+1}} \in \mathcal{G}\}$$

We say that $\mathcal{G}$ uniquely represents $X_{\text{con}}(\mathcal{G})$ if there is only one set of integers $\{k_i : i \in \mathbb{Z}\}$ satisfying the above property.

i.e. every point in X_{con} has a unique representation as a concatenation of generators.

A weaker condition: every finite word in X has at most one representation as a concatenation of generators (unique decipherability property).

Blanchard and Hansel (1986): every coded shift has a generating set $\mathcal{G}$ which has the unique decipherability property.

Béal, Perrin, Restivo (2023): For every coded shift X there exists a generating set $\mathcal{G}$ such that $X = X(\mathcal{G})$ and $\mathcal{G}$ uniquely represents $X_{\text{con}}(\mathcal{G})$.

Entropies

$\mathcal{G} \subset \mathcal{A}^*$ is fixed, $X = X(\mathcal{G})$, X_{con} consists of all concatenations of words in $\mathcal{G}$.

Define the residual set of X as $X_{\text{res}} = X \setminus X_{\text{con}}$ (points added under the closure).

Note that X_{con} and X_{res} are σ-invariant disjoint sets and $X = X_{\text{con}} \bigcup X_{\text{res}}$.

Our approach is to study the entropy on the sets X_{con} and X_{res} separately.

If μ is an ergodic measure on X then either $\mu(X_{\text{con}}) = 1$ or $\mu(X_{\text{res}}) = 1$

We define the concatenation entropy of X by

$$h_{\text{con}}(X) = \sup\{h_\sigma(\mu) : \mu \text{ is } \sigma\text{-invariant and } \mu(X_{\text{con}}) = 1\}$$

and the residual entropy of X by

$$h_{\text{res}}(X) = \sup\{h_\sigma(\mu) : \mu \text{ is } \sigma\text{-invariant and } \mu(X_{\text{res}}) = 1\}$$

Since $h_{\text{top}}(X) = \sup_\mu h_\sigma(\mu)$ (Variational Principle) we get

$$h_{\text{top}}(X) = \max\{h_{\text{con}}(X), h_{\text{res}}(X)\}$$

Flexibility of the Residual Set

Theorem (K., Schmoll, Wolf, 2024)

Let Z be any transitive subshift on a finite alphabet $\mathcal{A}$ which is not the full shift, and let $\epsilon > 0$. Then there exists a generating set $\mathcal{G}$ on the same alphabet $\mathcal{A}$ such that for $X = X(\mathcal{G})$ the following holds:

- *$\mathcal{G}$ uniquely represents X_{con};*
- *$Z \subset X_{\mathrm{res}}$, and if μ is σ-invariant with $\mu(X_{\mathrm{res}}) = 1$ then $\mu(Z) = 1$;*
- *$h_{\mathrm{con}}(X) < \epsilon$.*

In particular, if Z has multiple measures of maximal entropy and $h_{\mathrm{top}}(Z) > \epsilon$ then X has multiple measures of maximal entropy (the same as Z).

For examples of subshifts with various cardinalities of measures of maximal entropy see Buzzi, *Subshifts of quasi-finite type*, Inventiones(2005)

Uniqueness of the measures of maximal entropy doesn't hold for $h_{\mathrm{con}}(X) < h_{\mathrm{res}}(X)$.

Uniqueness of Measures of Maximal Entropy

Theorem (K., Schmoll, Wolf, 2024)

Let $X = X(\mathcal{G})$ be a coded shift such that $\mathcal{G}$ uniquely represents X_{con} and $h_{\mathrm{con}}(X) > h_{\mathrm{res}}(X)$. Then X has a unique measure of maximal entropy. Moreover, $h_{\mathrm{top}}(X) = \log \lambda$ where $\lambda > 0$ is the unique solution of the equation

$$\sum_{g \in \mathcal{G}} \lambda^{-|g|} = 1.$$

Examples (on $\mathcal{A} = \{0, 1\}$):

1. $\mathcal{G} = \{0, 01\}$, $X(\mathcal{G})$ is the golden mean shift , then $h_{\mathrm{top}}(X) = \log \lambda$, where $\lambda^{-1} + \lambda^{-2} = 1 \quad \Leftrightarrow \quad \lambda^2 - \lambda - 1 = 0 \quad \Leftrightarrow \quad \lambda = \frac{1+\sqrt{5}}{2}$
2. $\mathcal{G} = \{1, 00\}$, $X(\mathcal{G})$ is the even shift , then $h_{\mathrm{top}}(X) = \log \lambda$, where $\lambda^{-1} + \lambda^{-2} = 1 \quad \Leftrightarrow \quad \lambda^2 - \lambda - 1 = 0 \quad \Leftrightarrow \quad \lambda = \frac{1+\sqrt{5}}{2}$

The even shift and the golden mean shift have the same entropy because they are coded with generators having the same number of words of exactly the same length.

Uniqueness of Measures of Maximal Entropy

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History of the entropy formula:

- It first appeared in the book of Lind and Marcus (1989), as an exercise, in the case of S-gap shifts.
- Spandl published a proof (still for S-gap shifts) in 2007, but it is incomplete.
- Climenhaga presented two proofs for S-gap shifts in his blog in 2017.
- In 2019 García-Ramos and Pavlov published a fixed Spandl's proof and extended it for a certain class of coded shifts.
- Here we prove it in full generality.

Equilibrium States

Let (X, σ) be a coded shift and $\phi : X \to \mathbb{R}$ be a continuous potential.

The topological pressure of ϕ is defined by $P_{\text{top}}(\phi) = \sup_\mu\{\mathsf{h}_\sigma(\mu) + \int \phi \, d\mu\}$, where μ runs over the set of all σ-invariant probability measures on X
A measure μ which realizes the supremum is called an equilibrium state of ϕ.

Since the system is compact and the map $\mu \mapsto h_\mu$ is upper semi-continuous, there exists at least one equilibrium state. The main question is uniqueness.

We define the concatenation pressure of ϕ by

$$P_{\text{con}}(\phi) = \sup\{h_\sigma(\mu) + \int \phi \, d\mu : \mu \text{ is } \sigma\text{-invariant and } \mu(X_{\text{con}}) = 1\}$$

and the residual pressure of ϕ by

$$P_{\text{res}}(\phi) = \sup\{h_\sigma(\mu) + \int \phi \, d\mu : \mu \text{ is } \sigma\text{-invariant and } \mu(X_{\text{res}}) = 1\}$$

Then $P_{\text{top}}(\phi) = \max\{P_{\text{con}}(\phi), \, P_{\text{res}}(\phi)\}$.

Uniqueness of Equilibrium States

In [K., Schmoll, Wolf, 2024] we show that the equilibrium state of a Hölder potential $\phi : X(\mathcal{G}) \to \mathbb{R}$ is unique provided that

- $P_{\mathrm{con}}(\phi) > P_{\mathrm{res}}(\phi)$;
- $\limsup_{n\to\infty} \frac{1}{n} \log \mathrm{card}\{g \in \mathcal{G} : |g| = n\} = 0$
- every equilibrium state of ϕ has non-zero entropy.

The first condition is essential. We are able to remove the other two conditions.

Theorem (K. and Urbanski, 2025)

Let $X = X(\mathcal{G})$ be a coded shift such that $\mathcal{G}$ uniquely represents X_{con} and let $\phi : X \to \mathbb{R}$ be a Hölder potential such that $P_{\mathrm{con}}(\phi) > P_{\mathrm{res}}(\phi)$. Then ϕ has a unique equilibrium state.

Comparison to Previous Results

Climenhaga (2018), Pavlov (2020): uniqueness results for the measure of maximal entropy of a coded shift X under the assumption $h_{\text{top}}(X) > h_{\text{top}}(X_{\lim})$, where

$$X_{\text{lim}} = \{x \in X : \forall n \in \mathbb{N}\ \exists g(n) \in \mathcal{G},\ x_{-n}, ..., x_n] \subset g(n)\}.$$

(Similar results also hold for equilibrium states)

Since $h_{\text{res}}(X) \leq h_{\text{top}}(X_{\lim}) < h_{\text{top}}(X) = h_{\text{con}}(X)$, our theorem also gives uniqueness.

Theorem (K., Schmoll, Wolf, 2024)

Let X be a coded shift so that $\mathcal{G}$ uniquely represents X_{con} and $h_{\text{top}}(X) > h_{\text{top}}(X_{\lim})$. Given $\epsilon > 0$ there is a generating set $\widetilde{\mathcal{G}}$ such that $\widetilde{X} = X(\widetilde{\mathcal{G}}) \supset X$ and

1. $\widetilde{X} = \widetilde{X}_{\lim}$;
2. $h_{\text{res}}(X) = h_{\text{res}}(\widetilde{X}) < h_{\text{con}}(\widetilde{X})$;
3. $h_{\text{top}}(X) < h_{\text{top}}(\widetilde{X}) < h_{\text{top}}(X) + \epsilon$.

In particular, the Climenhaga/Pavlov results can not be applied to $\widetilde{\mathcal{G}}$.

Properties of the Measure of Maximal Entropy

We provide a simple explicit description of the measure of maximal entropy.

Let X be a coded shift space with a generating set $\mathcal{G}$. For $g \in \mathcal{G}$ we define

$$[g] = \{x \in X : x = \cdots x_{-1}.x_0x_1\cdots \text{ with } x_i \in \mathcal{A} \text{ and } x_0x_1 \ldots x_{|g|-1} = g\};$$
$$[\![g]\!] = \{x \in X : x = \cdots g_{-2}g_{-1}.g_0g_1g_2\cdots \text{ with } g_i \in \mathcal{G} \text{ and } g_0 = g\}$$

$[g]$ is the standard cylinder set of the word g in X.

$[\![g]\!]$ is a set of all concatenations with g appearing at zeroth coordinate.

Similarly, $[\![g_0\cdots g_k]\!] = \{x = \cdots g'_{-1}.g'_0g'_1\cdots : g'_i \in \mathcal{G} \text{ and } g'_j = g_j \text{ for } j = 0, \ldots, k\}$.

Since $[\![g_0\cdots g_k]\!]$ depends on a choice of $\mathcal{G}$ for X we call such sets $\mathcal{G}$-cylinders.

One can show that

- $\mathcal{G}$-cylinders are Borel sets.
- For any $w \in \mathcal{L}(X)$, $[w] \cap X_{\text{con}}$ is a disjoint union of shifted $\mathcal{G}$-cylinders.
- Any measure μ with $\mu(X_{\text{con}}) = 1$ is determined by its mass on $\mathcal{G}$-cylinders.

Properties of the Measure of Maximal Entropy

We say that a measure μ is $\mathcal{G}$-Bernoulli if there exist $p_g > 0,\ g \in \mathcal{G}$ with $\sum_{g\in\mathcal{G}} p_g = 1$ and $c > 0$ such that for any $k \in \mathbb{N}$ and any $g_0, \ldots, g_k \in \mathcal{G}$ we have

$$\mu([\![g_0 \cdots g_k]\!]) = \frac{1}{c} p_{g_0} \cdots p_{g_k}.$$

Theorem (K., Schmoll, Wolf, 2024)

Let $X = X(\mathcal{G})$ be a coded shift such that $\mathcal{G}$ uniquely represents X_{con} and $h_{\mathrm{con}}(X) > h_{\mathrm{res}}(X)$. Then the unique measure of maximal entropy $\mu_{\max}$ is $\mathcal{G}$-Bernoulli with $p_g = \exp(-|g| h_{\mathrm{top}}(X))$ and $c = \sum_{g\in\mathcal{G}} |g| \exp(-|g| h_{\mathrm{top}}(X))$.

Corollary: Suppose a coded shift X has a measure of maximal entropy which assigns full measure to a subshift $Y \subset X$ with $Y \neq X$. Then $h_{\mathrm{con}}(X) \leq h_{\mathrm{res}}(X)$ for any generating set $\mathcal{G}$ with $X = X(\mathcal{G})$, and which uniquely represents $X_{\mathrm{con}}(\mathcal{G})$.

The Dyck Shift

The Dyck shift is the go-to example of a coded shift where the assumptions of the uniqueness theorems fail: for its canonical generating set we have $X = X_{\text{lim}}$ and $h_{\text{con}}(X) < h_{\text{res}}(X)$. Also, the Dyck shift does have two ergodic measures of maximal entropy.

The Dyck shift is the smallest shift space on $\mathcal{A} = \{(,), [,]\}$ which contains the set of bi-infinite sequences where every (must be closed by) and every [must be closed by].

Krieger (1974): Dyck shift has topological entropy $\log 3$ and admits exactly two ergodic measures of maximal entropy which are fully supported.

The Dyck shift X is a coded with $\mathcal{G} = \bigcup W_n$ where the sets W_n are recursively defined by $W_1 = \{(\,), [\,]\}$, $W_n = \{(w), [w] : w \in (\cup_{i=1}^{n-1} W_i)^* \text{ and } |w| = 2(n-1)\}$.
Then $\mathcal{G}$ uniquely represents $X_{\text{con}}(\mathcal{G})$, $X_{\text{lim}}(\mathcal{G}) = X(\mathcal{G})$, and $h_{\text{con}}(\mathcal{G}) < h_{\text{res}}(\mathcal{G})$.

We show: There exist $\mathcal{G}_1$ and $\mathcal{G}_2$ such that $X = X(\mathcal{G}_i)$, $h_{\text{con}}(\mathcal{G}_i) = h_{\text{res}}(\mathcal{G}_i)$ and the two ergodic measures of maximal entropy of X are $\mathcal{G}_1$- respectively $\mathcal{G}_2$-Bernoulli.

Case $h_{\text{con}}(X) = h_{\text{res}}(X)$

Suppose $X = X(\mathcal{G})$, $\mathcal{G}$ uniquely represents X_{con}, and $h_{\text{con}}(X) = h_{\text{res}}(X)$.

We have examples where X has multiple measures of maximal entropy, one on $X_{\text{con}}(\mathcal{G})$ and the others on $X_{\text{res}}(\mathcal{G})$.

We know that there could be only one measure of maximal entropy on $X_{\text{con}}(\mathcal{G})$.

Questions:

1. If $h_{\text{con}}(X) = h_{\text{res}}(X)$, does there always exists measure of maximal entropy which assigns full measure to X_{con}?
2. If $h_{\text{con}}(X) = h_{\text{res}}(X)$, can the measure of maximal entropy be unique?
3. Does there exists a coded shift $X(\mathcal{G})$ such that $\mathcal{G}$ uniquely represents $X_{\text{con}}(\mathcal{G})$ and $\limsup_{n\to\infty} \frac{1}{n} \log |\{g \in \mathcal{G} : |g| = n\}| = h_{\text{top}}(X)$?

Thank you!