Flexibility of the Pressure Function

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Flexibility Program

Katok's Flexibility Program [Bochi, Katok, Rodriguez Hertz (2019)]:

to understand the most general constraints which define a common class of systems and build tools to change dynamical specifications within those constraints.

"There should be no restrictions on dynamical characteristics apart from a few obvious ones."

- Erchenko, Katok (2019): *Flexibility of entropies for surfaces of negative curvature.*
- Alsedà, Misiurewicz, Pérez (2020): *Flexibility of entropies for piecewise expanding unimodal maps.*
- Carrasco, Saghin (2021): *Extended flexibility of Lyapunov exponents for Anosov diffeomorphisms.*
- Banerjee, Kunde, and Wei(2021): *Flexibility of the slow entropy for rigid transformations.*
- We obtain the flexibility of the pressure function for compact symbolic systems

The Pressure Function

(Σ, T) is a shift over a finite alphabet. $\phi : \Sigma \to \mathbb{R}$ is a continuous potential.

The topological pressure of ϕ is defined by $P_{\text{top}}(\phi) = \sup_{\mu \in \mathcal{M}} \{\mathsf{h}_\mu + \int \phi \, d\mu\}$,
where $\mathcal{M}$ is the set of all T-invariant probability measures and
h_μ is the measure-theoretic entropy of μ.
A measure $\mu \in \mathcal{M}$ which realizes the supremum is called an equilibrium state of ϕ

Fix m continuous potentials $\phi_1, ..., \phi_m$.
For $(t_1, ...t_m) \in \mathbb{R}^m$ the multivariable pressure function is the map

$$(t_1, ..., t_m) \longmapsto P_{\text{top}}(t_1\phi_1 + ... + t_m\phi_m).$$

In multifractal analysis such pressure function is used as the main tool to compute the dimension spectra of the simultaneous level sets.

Properties of the Pressure

"Obvious" properties of the pressure function:

- Variational Principle $\Longrightarrow$ it is convex and Lipschitz
- convexity $\Longrightarrow$ there is a supporting hyperplane at each point of its graph
- the equilibrium states are tangent functionals to the pressure $\Longrightarrow$ the vertical intercept of the hyperplane is the entropy of an equilibrium state (must be between 0 and $h_{\text{top}}(\Sigma)$)

We show that these conditions are the **necessary** and **sufficient**.

The Main Result

Theorem 1

Let $\alpha > 0$ and let $F(t_1, \ldots, t_m)$ be a convex Lipschitz function on $(\alpha, \infty)^m$ such that all the supporting hyperplanes to the graph of F intersect the vertical axis in a closed interval $[b, c] \subset [0, \infty)$. Then there exists a full shift on a finite alphabet and continuous potentials $\phi_1, \ldots, \phi_m$ such that $P_{\text{top}}(t_1\phi_1 + \ldots + t_m\phi_m) = F(t_1, \ldots, t_m)$ for all $(t_1, \ldots, t_m) \in (\alpha, \infty)^m$.

Our proof is explicit and constructive: For an arbitrary function F satisfying these properties we build a set of m continuous potentials whose pressure function coincides with F on $(\alpha, \infty)^m$.

This result falls in line with the flexibility program: Within the class of full shifts on finite alphabets we identify the general constraints on the pressure function and provide a tool to acquire any pressure function within those constraints.

Domain of F

Among symbolic systems our focus is the full shifts.

$$P_{\text{top}}(\phi) = \sup_{\mu \in \mathcal{M}} \{\mathsf{h}_\mu + \int \phi \, d\mu\} \implies P_{\text{top}}(0) = h_{\text{top}}(\Sigma)$$

$(0, ..., 0) \in \text{Domain}(F) \implies$

- $F(0, ..., 0) = \log d$ for some integer $d \geq 2$ (an additional restriction on F)
- Σ must be the full shift on d symbols (a fixed dynamical system)

To operate freely within the class of full shifts the domain of F should not contain the origin in its convex hull.

We chose $\text{Domain}(F) = (\alpha, \infty)^m$.

(for $m = 1$ the positive parameter t has a physical interpretation as the inverse temperature.)

Equilibrium States

Uniqueness equilibrium states vs. regularity of the pressure:

Walters(1992):

$P_{\text{top}}(.)$ is Gateaux differentiable at ϕ $\iff$ ϕ has a unique equilibrium state.

Non-differentiability of the pressure function $t \mapsto P(t\phi)$ at t $\Longrightarrow$ non-uniqueness of equilibrium states for $t\phi$

"$\Longleftarrow$" is not true

Leplaideur (2015) gave an example of a continuous ϕ on a mixing subshift of finite type such that $P_{\text{top}}(t\phi)$ is analytic, but there are two values of t for which $t\phi$ has two equilibrium states.

We show that at any smooth point of the pressure function the cardinality of the set of ergodic equilibrium states for the potential may be any number, finite or infinite.

Cardinality of Equilibrium States

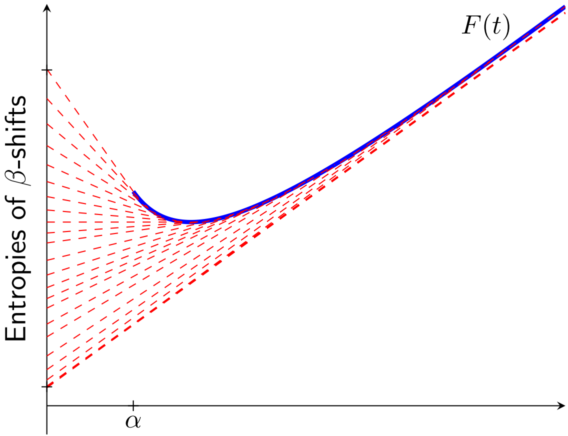
The next theorem provides a flexible way of constructing systems of potentials with varying cardinalities of the equilibrium measures.

Theorem 2

Let $f(t)$ be a strictly convex differentiable function on (α, ∞) with support line intercepts lying in a bounded interval $[b, c] \subset [0, \infty)$. Then for any $\ell \in \mathbb{N}$ and any upper semi-continuous function $N\colon (\alpha, \infty) \to \{1, \ldots, \ell, \infty\}$, there exists a full shift on a finite alphabet and a continuous potential function ϕ such that

- $P(t\phi) = f(t)$ *for all* $t \in (\alpha, \infty)$;
- *the cardinality of the set of ergodic equilibrium states for* $t\phi$ *is exactly* $N(t)$.

General Idea of the Proof



- Fix a convex $F(t)$ on (α, ∞).
- $F(t)$ has a supporting line through each point on its graph
- Define ϕ so that the equilibrium state μ_t of $t\phi$ satisfies
  - h_{μ_t} = the y-intercept of the supporting line at t
  - $\int \phi \, d\mu_t$ = the slope of supporting line at t

General Idea of the Proof

For $\beta > 1$ the β-shift X_β consists of the sequences of the coefficients in the expansions of reals in base β.

Good properties:

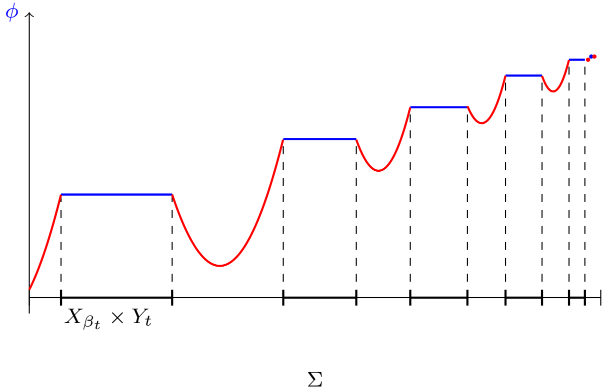
- $\{X_\beta : \beta > 1\}$ is a family of shift-invariant closed sets
- $h_{\text{top}}(X_\beta) = \log \beta$
- X_β has a unique measure of maximal entropy

Obstacle: β-shifts are nested.

Solution: Take a product of each X_β with a suitably chosen Sturmian shift Y.

Sturmian shifts are very low complexity systems and do not contribute to the entropy. We define ϕ on $X_{\beta_t} \times Y_t$ to be the slope of the corresponding supporting line to $F(t)$. Make ϕ drop sharply outside $\bigcup X_{\beta_t} \times Y_t$ and force the equilibrium measures at all values of t to be supported on $\bigcup X_{\beta_t} \times Y_t$.

Visual Aid



Issues:

- Continuity of ϕ
- Estimates on the pressure (!)

The main difficulty is to ensure that the drop-off is sufficiently steep so that for any ergodic μ not supported on $\bigcup X_{\beta_t} \times Y_t$ we have $h_\mu + t\int \phi \, d\mu < P_{\text{top}}(t\phi|_{X_{\beta_t} \times Y_t})$.

Our Technique

Our Technique:

- For each $x \in \Sigma$ we look for blocks within x from $X_{\beta_t} \times Y_t$s
- We note their locations and sizes.
- To store this data we introduce an additional subshift $Z \subset \{0,1\}^{\mathbb{Z}}$ and consider $\Sigma \times Z$.
  We call Z the pin-sequence space since for a pair (x, z) a 1 in z pins exactly the place in x where one block from $\bigcup X_{\beta_t} \times Y_t$ ends and another one begins.
- We define $\phi(x)$ based on the information from Z.
- All the estimates on the pressure are performed on $\Sigma \times Z$ and then projected back to Σ.

Phase Transitions

The one-parameter pressure function $t \mapsto P(t\phi)$ is of particular interest since $t > 0$ can be interpreted as the inverse temperature of the system.

From the statistical physics point of view, $P_{\text{top}}(\phi)$ corresponds to the minimum of the free energy $E_\mu = -(\mathsf{h}_\mu + \int \phi \, d\mu)$. An equilibrium state μ minimizes the free energy.

When the temperature changes, the equilibrium of the system changes as well.
A phase transition refers to a qualitative change of the properties of a dynamical system as a result of the change in temperature.
Intuitively, this means co-existance of several equilibria at the same temperature.

We are interested in the values of t for which $t\phi$ has more than one equilibrium state.

$P_{\text{top}}(t\phi)$ in not differentiable at t_0 $\iff$ $t_0\phi$ has two equilibrium states with distinct entropies.

Lack of Phase Transitions

The potential ϕ has a phase transition at t_0 if the pressure function $t \mapsto P_{\text{top}}(t\phi)$ is not differentiable at t_0 (first order phase transition).

Ruelle (1968): If Σ is a transitive subshift of finite type then the pressure functional P_{top} acts real analytically on the space of Hölder continuous potentials.

In particular, when ϕ is Hölder

- the pressure function $t \mapsto P_{\text{top}}(t\phi)$ is analytic,
- $t\phi$ has a unique equilibrium state for any t,

and hence there are no phase transitions.

In order to allow the possibility of phase transitions one needs to consider potential functions that are merely continuous.

Multiple Phase Transitions

Sarig provided examples of infinitely many phase transitions for Markov shifts on a countable alphabet. For shifts on a finite alphabet there were no examples in the literature with more than two phase transitions.

K., Quas, Wolf (2020): Let X be a two-sided full shift on two symbols. Then for any given $\alpha > 0$ and any increasing sequence of positive real numbers $(t_n) \subset (\alpha, \infty)$ there is a continuous potential $\phi : X \to \mathbb{R}$ which has phase transitions precisely at t_n.

$t \mapsto P_{\text{top}}(t\phi)$ is convex $\implies$ at most countable points of non-differentiability.
A convex function may have a countable dense set of points of non-differentiability.
Is it feasible for ϕ to have a dense set of phase transitions?

Corollary[K., Quas (2021)]

For any given countable set $S \subset (\alpha, \infty)$ there is a continuous potential ϕ whose phase transitions in (α, ∞) occur precisely at points in S.

Application in Multifractal Analysis

Barreira, Saussol and Schmeling (2002); Climenhaga (2014):

For continuous potentials $\phi_1, ..., \phi_m$ and $\psi_1, ..., \psi_m$ with $\psi_i > 0$

$$h_{\text{top}} K(\alpha) = \inf \left\{ P\left(\sum_{i=1}^{m} t_i(\phi_i - \alpha_i \psi_i) \right) : (t_1, ..., t_m) \in \mathbb{R}^m \right\}$$

where $K(\alpha) = \left\{ x \in \Sigma : \lim_{n\to\infty} \frac{\phi_i(x)+\phi_i(Tx)+...+\phi_i(T^n x)}{\psi_i(x)+\psi_i(Tx)+...+\psi_i(T^n x)} = \alpha_i \text{ for all } i \right\}$

Question 1

Can we obtain families of potentials with "interesting" properties of the entropy spectra?

Pressure Functions on a Fixed Shift

In Theorem 1 we have identified necessary and sufficient conditions under which a function $F\colon (\alpha,\infty)^m \to \mathbb{R}$ may be represented in the form $P_{\mathrm{top}}(t_1\phi_1 + \ldots + t_m\phi_m)$ for continuous functions $\phi_1, \ldots, \phi_m$ defined on a full shift.

Can necessary and sufficient conditions for such a representation be identified for functions $F\colon \mathbb{R}^m \to \mathbb{R}$?
(An additional restriction on the value of F at the origin is needed in this case and this value determines the full shift.)

Question 2

Let F be a Lipschitz convex function on $\mathbb{R}^m$ whose supporting hyperplane intercepts lie in a bounded sub-interval of $\mathbb{R}_{\geq 0}$ such that $F(0,\ldots,0) = \log d$ for some integer $d \geq 2$. Do there exist continuous functions $\phi_1, \ldots, \phi_m$ on the full shift on d-symbols such that $F(t_1,\ldots,t_m) = P(t_1\phi_1 + \ldots + t_m\phi_m)$ for all $(t_1,\ldots,t_m) \in \mathbb{R}^m$?

Hölder Potentials

If the potentials $\phi_1, ..., \phi_m$ are Hölder the pressure function $P_{\text{top}}(t_1\phi_1 + ... + t_m\phi_m)$ is analytic.

Starting with an analytic function $F(t_1, ..., t_m)$ we obtain from Theorem 1 a set of continuous potentials for which the pressure function coincides with F.
(Our potentials are not Hölder.)

Would any analytic convex function be a pressure function for a set of Hölder continuous potentials?

Question 3

Is it possible to give a nice characterization of those functions $F\colon \mathbb{R}^m \to \mathbb{R}$ that arise as $P_{\text{top}}(t_1\phi_1 + \ldots + t_m\phi_m)$ where $\phi_1, \ldots, \phi_m$ are required to be Hölder continuous?

Flexibility of Equilibrium States

Israel (1979): For any full shift and any subset K of the set of the shift-invariant Borel probability measures that is the weak*-closure of the linear span of a non-empty collection of ergodic measures, there exists a potential ϕ whose equilibrium states are precisely K.

Can we specify the sets of equilibrium states at various points of the pressure function?

Question 4

Let $0 < t_1 < \ldots < t_n$ and let $p_1, \ldots, p_n$ be such that there is a convex function passing through $(t_j, p_j)_{j=1}^n$. Let $K_1, \ldots, K_n$ be disjoint weak*-closures of linear spans of collections of ergodic invariant measures. Does there exist a continuous potential ϕ such that $P(t_j\phi) = p_j$ for $j = 1, \ldots, n$ and such that the equilibrium states of $t_j\phi$ are precisely K_j?

Thank you!