Flexibility of the Pressure Function

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Flexibility Program

Katok's Flexibility Program [Bochi, Katok, Rodriguez Hertz (2019)]:

to understand the most general constraints which define a common class of systems and build tools to change dynamical specifications within those constraints.

"There should be no restrictions on dynamical characteristics apart from a few obvious ones."

- Erchenko, Katok (2019): *Flexibility of entropies for surfaces of negative curvature.*
- Alsedà, Misiurewicz, Pérez (2020): *Flexibility of entropies for piecewise expanding unimodal maps.*
- Carrasco, Saghin (2021): *Extended flexibility of Lyapunov exponents for Anosov diffeomorphisms.*
- Roth, Snoha (Previous Talk!): *Flexibility of the polynomial entropy for homeomorphisms.*
- We obtain the flexibility of the pressure function for compact symbolic systems

The Pressure Function

Let $\phi : X \to \mathbb{R}$ be a continuous potential associated with a symbolic dynamical system (X, T) over a finite alphabet.

The topological pressure of ϕ is defined by $P_{\text{top}}(\phi) = \sup_{\mu \in \mathcal{M}} \{\mathsf{h}_\mu + \int \phi \, d\mu\}$,
where $\mathcal{M}$ is the set of all T-invariant probability measures and h_μ is the measure-theoretic entropy of μ.

We introduce a parameter $t > 0$ and study the pressure function

$$t \mapsto P_{\text{top}}(t\phi).$$

"Obvious" Properties:

The pressure function is convex, Lipschitz, and has an asymptote as $t \to \infty$.

We show that these are the **only** restrictions.

The Main Result

Theorem

Let $f(t)$ be a convex function on (α, ∞) with $\alpha > 0$ such that all the supporting lines to the graph of f intersect the vertical axis in a closed interval $[b, c] \subset [0, \infty)$. Then there exists a full shift on a finite alphabet and a continuous potential ϕ such that $P_{\text{top}}(t\phi) = f(t)$ for all $t \in (\alpha, \infty)$.

Remarks:

- boundedness of intercepts $\Longleftrightarrow$ Lipschitz $+$ slant asymptote
- we have full control over the pressure function over (α, ∞), but not near $t = 0$
- we develop a method to explicitly construct the potential ϕ.

We identify the general constraints on the pressure function and provide a tool to acquire any pressure function within those constraints.

Phase Transitions

From the statistical physics point of view, $P_{\text{top}}(\phi)$ corresponds to the minimum of the free energy $E_\mu = -(\mathsf{h}_\mu + \int \phi\, d\mu)$.

A measure $\mu \in \mathcal{M}$ which minimizes the free energy (i.e. $P_{\text{top}}(\phi) = \mathsf{h}_\mu + \int \phi\, d\mu$) is called an equilibrium state of ϕ.

We study the equilibrium states of the potential $t\phi$ where the parameter $t > 0$ is interpreted as the inverse temperature of the system.

When the temperature changes, the equilibrium of the system changes as well. A phase transition refers to a qualitative change of the properties of a dynamical system as a result of the change in temperature. Intuitively, this means co-existance of several equilibria at the same temperature.

We are interested in the values of t for which potential $t\phi$ has more than one equilibrium state.

Connection to Pressure Function

Co-existence of several equilibria vs. regularity of the pressure:

- P_{top} is Gateaux differentiable at $\phi \iff \phi$ has a unique equilibrium state
- If the pressure function $t \mapsto P_{\text{top}}(t\phi)$ is not differentiable at t_0 then $t_0\phi$ has at least two equilibrium states.
- Non-uniqueness of equilibrium states for $t_0\phi$ does not imply non-differentiability of $P_{\text{top}}(t\phi)$ at t_0.
  Leplaideur (2015): there is a continuous ϕ on a mixing subshift of finite type such that $P_{\text{top}}(t\phi)$ is analytic on some interval, but uniqueness of equilibrium states fails for two distinct values of t in that interval.
- $P_{\text{top}}(t\phi)$ in not differentiable at $t_0 \iff t_0\phi$ has two equilibrium states with distinct entropies.

Lack of Phase Transitions

The potential ϕ has a phase transition at t_0 if the pressure function $t \mapsto P_{\text{top}}(t\phi)$ is not differentiable at t_0 (first order phase transition).

Ruelle (1968): If X is a transitive subshift of finite type then the pressure functional P_{top} acts real analytically on the space of Hölder continuous potentials.

In particular, when ϕ is Hölder

- the pressure function $t \mapsto P_{\text{top}}(t\phi)$ is analytic,
- $t\phi$ has a unique equilibrium state for any t,

and hence there are no phase transitions.

In order to allow the possibility of phase transitions one needs to consider potential functions that are merely continuous.

Multiple Phase Transitions

Sarig provided examples of infinitely many phase transitions for Markov shifts on a countable alphabet. For shifts on a finite alphabet there were no examples in the literature with more than two phase transitions.

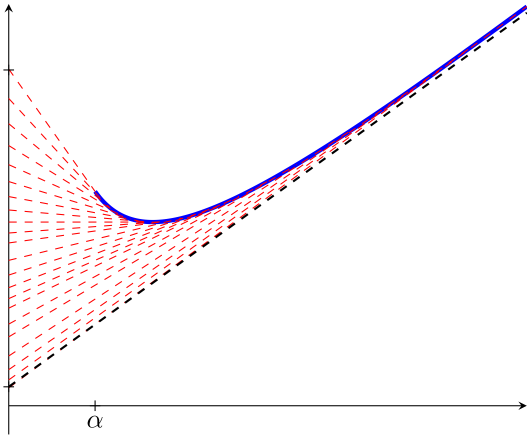
K., Quas, Wolf (2020): *Let X be a two-sided full shift on two symbols. Then for any given $\alpha > 0$ and any increasing sequence of positive real numbers $(t_n) \subset (\alpha, \infty)$ there is a continuous potential $\phi : X \to \mathbb{R}$ which has phase transitions precisely at t_n.*

$t \mapsto P_{\text{top}}(t\phi)$ is convex $\implies$ at most countable points of non-differentiability.
A convex function may have a countable dense set of points of non-differentiability.
Is it feasible for ϕ to have a dense set of phase transitions?

Corollary[K., Quas (2021)]

For any given countable set $S \subset (\alpha, \infty)$ there is a continuous potential ϕ whose phase transitions in (α, ∞) occur precisely at points in S.

General Idea of the Proof



- Fix a convex $f(t)$ on (α, ∞) with a slant asymptote.
- $f(t)$ has a supporting line through each point on its graph
- Define ϕ so that the equilibrium state μ_t of $t\phi$ satisfies
  - h_{μ_t} is the y-intercept of the supporting line at t
  - $\int \phi \, d\mu_t$ is the slope of supporting line at t

Thank you!