Freezing Phase Transitions for Higher-Dimensional Subshifts

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Phase Transitions

A phase transition is observed when one follows an evolution of a system depending on continuous external factors and a sharp change of the behaviour of the system happens.

Fundamental goal in statistical physics: to understand the mechanism of this phenomenon.

Lattice models provide a mathematical framework to study how systems change between distinct states of matter (e.g., from a disordered phase to an ordered phase). These models are defined on a discrete lattice ($\mathbb{Z}^d$) and describe interactions between variables located at the lattice sites.

Thermodynamic formalism was largely developed by Ruelle to study phase transitions on lattice systems.

Lattice Systems

Let $X = \mathcal{A}^{\mathbb{Z}^d}$ be the d-dimensional full shift on a finite alphabet $\mathcal{A}$,
i.e., X is the space of all maps (configurations) $x : \mathbb{Z}^d \to \mathcal{A}$ with the product topology.

The shift map σ is the action of $\mathbb{Z}^d$ on X by translations: $(\sigma^{\boldsymbol{n}} x)_{\boldsymbol{m}} = x_{\boldsymbol{n}+\boldsymbol{m}}$.

For $x, y \in X$ define $\operatorname{dist}(x, y) = 2^{-n}$, where $n = n(x, y) = \min\{\|\boldsymbol{n}\|_\infty : x_{\boldsymbol{n}} \neq y_{\boldsymbol{n}}\}$
(compatible with the product topology).

A subshift X_0 is a closed subset of X which is $\sigma^{\boldsymbol{n}}$-invariant for every $\boldsymbol{n} \in \mathbb{Z}^d$.

$$\text{Topological entropy } h_{\text{top}}(X_0) = \lim_{n \to 0} \frac{\#\{x|_{[1,n]^d} : x \in X_0\}}{n^d}$$

Thermodynamic Formalism

Let $\phi : X \to \mathbb{R}$ be a continuous potential.

ϕ corresponds to an interaction potential in statistical physics, which is a family of functions indexed by finite subsets of $\mathbb{Z}^d$.

The topological pressure of ϕ is defined by

$$P_{\text{top}}(\phi) = \sup_{\mu \in \mathcal{M}} \{\mathsf{h}_\mu + \int \phi \, d\mu\},$$

where $\mathcal{M}$ is the set of all σ-invariant probability measures and h_μ is the measure-theoretic entropy of μ.

From the statistical physics point of view, $P_{\text{top}}(\phi)$ corresponds to the minimum of the free energy $E_\mu = -(\mathsf{h}_\mu + \int \phi \, d\mu)$.

A measure $\mu \in \mathcal{M}$ which minimizes the free energy (i.e. $P_{\text{top}}(\phi) = \mathsf{h}_\mu + \int \phi \, d\mu$) is called an equilibrium state of ϕ.

The Pressure Function

We introduce a parameter β (interpreted as the inverse temperature of the system) and study the equilibrium states of the potential $\beta\phi$.

Our main tool is the pressure function of ϕ, which is the map $p_\phi : \beta \mapsto P_{\text{top}}(\beta\phi)$.

$\beta = 0$ ("infinite temperature"): one equilibrium state which is the measure of maximal entropy on $\mathcal{A}^{\mathbb{Z}^d}$. This is the most disorderly phase possible, where ϕ plays no role.

$\beta \to \infty$ ("the zero temperature limit"): the weak*-accumulation points of equilibrium states $\mu_{\beta\phi}$ of $\beta\phi$ as $\beta \to \infty$.

Does $\lim \mu_{\beta\phi}$ exist as $\beta \to \infty$?

$d = 1$: **yes**, for locally constant potentials (finite-range interactions) [Brémont, 2008]
no, for Lipschitz potentials (exponentially decaying interactions) [Chazottes and Hochman, 2010]

$d \geq 2$: **no**, even for locally constant potentials

Maximizing Measures

$X = \mathcal{A}^{\mathbb{Z}^d}$, $\phi : X \to \mathbb{R}$ is a continuous potential, $p_\phi(\beta) = \sup\{h_\mu + \beta \int \phi \, d\mu\}$.

If μ_∞ is a weak*-accumulation point of equilibrium $\mu_{\beta\phi}$ of $\beta\phi$ as $\beta \to \infty$ then
$\int \phi \, d\mu_\infty = \max\left\{ \int \phi \, d\mu : \mu \in \mathcal{M} \right\}$ and $h_{\mu_\infty} = \max\left\{h_\mu : \mu \text{ maximizes } \int \phi \, d\mu\right\}$

The pressure function has a slant asymptote:
$p_\phi(\beta) = h_{\mu_{\beta\phi}} + \beta \int \phi \, d\mu_{\beta\phi} \underset{\text{large } \beta}{\approx} h_{\mu_\infty} + \beta \int \phi \, d\mu_\infty$, the asymptote is $h_{\mu_\infty} + \beta \int \phi \, d\mu_\infty$

If ϕ is not cohomologous to a constant and satisfies a natural "uniform summability" condition, the support of any of its maximizing measures is contained in a proper subshift $X_0 \subset X$ [Garibaldi and P. Thieullen, 2015].

If X_0 has a unique measure of maximal entropy, then the zero temperature limit exists.

Can equilibrium states collapse (become concentrated on X_0) at low but not zero temperature? Equivalently, can $p_\phi(\beta)$ reach its asymptote?

Freezing at low but non-zero temperature

Question 1: Given a potential ϕ, is it possible that the equilibrium states become the maximizing measures with maximal entropy as soon as β exceeds a certain critical value $\beta_c \in (0, +\infty)$?

Equilibrium states cease to change with temperature and the system is "frozen".

Question 2: Given a subshift X_0, can we construct a function ϕ whose equilibrium states coincide, for all $\beta > \beta_c$, with the measures of maximal entropy of X_0?

The subshift can model a crystal (finitely many configurations, each being a shift of another), or a quasicrystal (a uniquely ergodic, zero-entropy subshift with infinitely many configurations, and thus no periodic configurations).

Freezing on a Subshift

Let X_0 be a proper subshift of X.

A continuous $\phi\colon X \to \mathbb{R}$ is said to freeze on X_0 if there is $\beta_c \in (0, \infty)$ such that

- for $\beta > \beta_c$ the equilibrium states for $\beta\phi$ are supported on X_0;
- for $\beta < \beta_c$ no equilibrium state for $\beta\phi$ is supported on X_0.

The equilibrium states of ϕ and $\phi + c$ are the same $\Rightarrow$ assume $\max_\mu\{\int \phi \, d\mu\} = 0$.

Then the pressure function of ϕ looks like this:

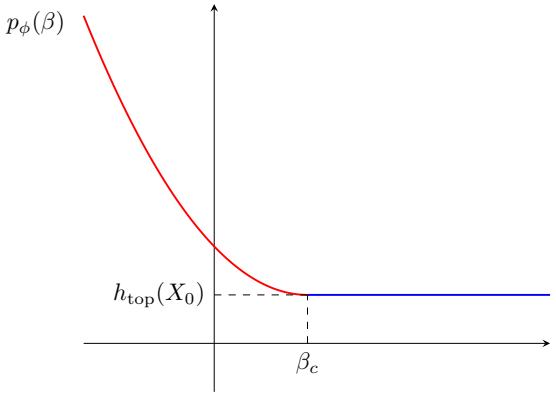

Figure: Phase transition at β_c.

Existence of Freezing Phase Transitions

Theorem A

For any proper subshift X_0 of a full d-dimensional shift X on a finite alphabet, there is an explicit continuous potential ϕ that freezes on X_0 at some $\beta_c > 0$. Moreover, for $\beta > \beta_c$, the equilibrium states of $\beta\phi$ are the measures of maximal entropy of X_0.

We have explicit bounds on $\operatorname{var}_n(\phi) = \sup\{|\phi(x) - \phi(y)| : x, y \in X, \operatorname{dist}(x, y) \leq 2^{-n}\}$.

In particular, when the subshift X_0 is a full shift on $\mathcal{A}' \subsetneq \mathcal{A}$, $\operatorname{var}_n \phi = O\left(\frac{\log\log n}{n^d}\right)$.

Theorem B

Let X be a full d-dimensional shift on a finite alphabet and $\phi : X \to \mathbb{R}$ be a continuous potential. If $\sum_n n^{d-1} \operatorname{var}_n \phi < \infty$ then ϕ cannot have a freezing phase transition.

A freezing phase transition cannot occur for potentials ϕ such that $\operatorname{var}_n \phi \leq \frac{1}{n^{d+\epsilon}}$.

Sufficient Conditions for Freezing

Sufficient conditions for a potential ϕ to freeze on X_0:

1. $\phi(x) = 0$ for $x \in X_0$ and $\phi(x) < 0$ for any $x \notin X_0$;
2. $P_{\text{top}}(\phi) = h_{\text{top}}(X_0)$;
3. The set of equilibrium states for ϕ is exactly the set of measures of maximal entropy for X_0.

Then ϕ freezes on X_0 at $\beta_c \in (0, 1]$, where $\beta_c = \inf\{\beta : p_\phi(\beta) = h_{\text{top}}(X_0)\}$.

Define

$$\phi(x) = \begin{cases} 0 & \text{if} \quad x \in X_0, \\ -a_n & \text{if} \quad \text{dist}(x, X_0) = 2^{-n}, \end{cases}$$

where (a_n) is a decreasing sequence of positive numbers that converges to zero.

Clearly ϕ is continuous, vanishes on X_0, and negative off X_0.

We need to choose (a_n) to satisfy properties (2)-(3).

Elaboration on Theorem A

Theorem A

The potential ϕ defined above freezes on X_0 whenever

$$a_{2^{i+1}} \geq \left(\frac{\log n_i}{2^{id}} - h_{\text{top}}(X_0) \right) + \frac{2 \log i + 3}{2^{id}} \text{ for all } i \in \mathbb{N}_0,$$

where n_i denotes the number of admissible $2^i \times \cdots \times 2^i$ blocks in X_0.

Note that $\frac{\log n_i}{2^{id}} - h_{\text{top}}(X_0)$ is non-increasing and converges to 0.
If we define $a_j = a_{2^{i+1}}$ when $2^i < j \leq 2^{i+1}$, we obtain $\text{var}_n \phi = a_n$.
Therefore,

$$\text{var}_n \phi = \kappa_{\lfloor \log_2 n \rfloor} + O\left(\frac{\log \log n}{n^d} \right),$$

where $\kappa_i = \frac{1}{2^{id}} \log n_i - h_{\text{top}}(X_0)$ is the rate of convergence.

Constant-box $\mathbb{Z}^d$ Substitution

Let X_0 be a constant-box $\mathbb{Z}^d$ substitution dynamical system,
i.e. X_0 is obtained from a substitution rule where each symbol is mapped to a $m_1 \times m_2 \times \cdots \times m_d$ box.

One can compute the word complexity function of X_0:

$$\#\{x|_{[1,n]^d} : x \in X_0\} = O(n^{\log(m_1 \cdots m_d)/\log \min m_j}),$$

so that $\kappa_i = O(i/2^{id})$. We obtain

Corollary

For a constant-box $\mathbb{Z}^d$ substitution shift X_0 there is a potential ϕ with $\operatorname{var}_n \phi = O\left(\frac{\log n}{n^d}\right)$ such that ϕ has a freezing phase transition on X_0.

Previously Known Results

Previous studies on freezing phase transitions were focused on the case $d = 1$.

The constructions were carried out in one-sided settings, i.e. $X = \mathcal{A}^{\mathbb{N}}$.

Reasons:

- Ruelle's Perron-Frobenius operator can be used to evaluate the pressure.
- ϕ can be extended to $X = \mathcal{A}^{\mathbb{Z}}$ without altering the pressure function, so the results carry over (converse is not true).

[Walters, 2003]: necessary and sufficient conditions for a continuous two-sided potential to be cohomologous to a one-sided potential.

These conditions cannot be satisfied by any potential with a freezing phase transition.

General Existence

There is a very general result that guaranties the existence of ϕ.

K. and Thompson (2021)

Given a compact topological dynamical system (X, T) with positive entropy and upper semicontinuous entropy map, and any closed invariant positive entropy subset $X_0 \subset X$ there exist a continuous potential $\phi : X \to \mathbb{R}$ satisfying our sufficient conditions for freezing.

The proof is based on the theory of tangent functionals and relies on Israel's theorem.

It does not provide any information on the type of potentials which would generate a freezing phase transition.

One-dimensional One-sided Symbolic Settings

Define $\phi : X \to \mathbb{R}$ by $\phi(x) = -a_n$ if $\mathrm{dist}(x, X_0) = 2^{-n}$, where $a_n \searrow 0$.

[Hofbauer,1977]: $X = \{0,1\}^{\mathbb{N}}$, $X_0 = 0^{\infty}$, ϕ freezes if $\sum_{n\geq 0} e^{-(a_0+\cdots+a_n)} = 1$

[Olivier,1999]: $X = \{0,1,2\}^{\mathbb{N}}$, $X_0 = \{0,1\}^{\mathbb{N}}$, ϕ freezes at $\beta_c = 1$

[K. and Thompson, 2020]: $X = \mathcal{A}^{\mathbb{N}}$, X_0 is any SFT, ϕ freezes if $a_n = O(\frac{1}{\sqrt{n}})$

[Buzzi, Kloeckner and Leplaideur, 2023]: existence of freezing phase transitions for all zero-entropy subshifts (not constructive)

Thue-Morse subshift

Thue-Morse subshift is given by substitution $0 \to 01, 1 \to 10$.

It is uniquely ergodic and used to model quasicrystals

[Bruin and Leplaideur,2013]: Let X_0 be a Thue-Morse subshift and $\phi(x) = -a_n = -\frac{1}{n^a} + o(\frac{1}{n^a})$ if $\mathrm{dist}(x, X_0) = 2^{-n}$. Then ϕ has a freezing phase transition for $a \in (0, 1)$, no phase transitions for $a > 1$.

Q: Does ϕ have a phase transition for $a = 1$?

The proofs appear to be incomplete.

[Bedaride, Cassaigne, Hubert, Leplaideur, work in progress]: fix the proofs and establish freezing when $a_n \approx \frac{1}{n}$.

[Theorem A +Theorem B](two-sided settings): ϕ freezes when $a_n \approx \frac{\log n}{n}$ and does not freeze when $a_n \approx \frac{1}{n^{1+\epsilon}}$.

Happy Birthday Manfred!