Ground States on the Boundary of Rotation Sets

Tamara Kucherenko, CCNY
(joint work with Christian Wolf)

July 5, 2016

Equilibrium States

Let (X, d) be a compact metric space and $f : X \to X$ be a continuous map. Denote by $\mathcal{M}$ the set of all f-invariant probability measures.

We are interested in measures in $\mathcal{M}$ which maximize $h_\mu + \int \varphi \, d\mu$.

Here $\varphi : X \to \mathbb{R}$ is a continuous potential and h_μ is the measure-theoretic entropy of μ. The entropy h_μ measures the exponential growth rate of statistically significant orbits with respect to μ.

A measure $\mu_\varphi \in \mathcal{M}$ is an equilibrium state for φ if

$$\mathsf{h}_{\mu_\varphi} + \int \varphi \, d\mu_\varphi = \sup_{\nu \in \mathcal{M}} \{\mathsf{h}_\nu + \int \varphi \, d\nu\}.$$

If the map $\mu \mapsto h_\mu$ is upper semi-continuous, then there exists at least one equilibrium state. (True for subshifts of finite type)

Ground States and Zero Temperature Measures

We denote by $t = \frac{1}{T}$ the inverse temperature of the system and study the equilibrium states of the potentials $t\varphi$.

What is the limiting behavior of the set of equilibrium states of $t\varphi$ as $t \to \infty$?
(zero temperature case)

Suppose for all $t > 0$ the potential $t\varphi$ has a unique equilibrium state $\mu_{t\varphi}$.
Does the sequence $\{\mu_{t\varphi}\}$ converge in weak* topology?

If yes, the measure $\mu = \lim_{t\to\infty} \mu_{t\varphi}$ is called the zero temperature measure of φ.

The accumulation points of equilibrium states are called the ground states of φ.

Ground States and Zero Temperature Measures

Uniqueness of equilibrium states:

expansive homeomorphisms with specification + φ is Hölder
(subshifts of finite type, axiom A systems)

Even in the case of uniqueness, the zero temperature measures might not exist.

Chazottes, Hochman (2010): There is Lipschitz $\varphi : \Sigma_2 \to \mathbb{R}$ such that $\{\mu_{t\varphi}\}$ has two weak* accumulation points.

Convergence of $\{\mu_{t\varphi}\}$:

subshifts of finite type + φ depends on finitely many coordinates
(Bremont 2003)

The study concentrates on the properties of accumulation points of $\{\mu_{t\varphi}\}$
(ground states).

Ground States and Zero Temperature Measures

Jenkinson, Maulding, Urbanski ('05)
Morris ('05)
Leplaideur ('07)
$\Bigg\} \Longrightarrow$

Suppose X is a subshift of finite type,
$\varphi : X \to \mathbb{R}$ is Hölder and
μ is a ground state of φ. Then

- $\int \varphi \, d\mu = \max\{\int \varphi \, d\nu : \nu \in \mathcal{M}\}$
- $h_\mu = \lim_{t\to\infty} h_{\mu_{t\varphi}}$
- $h_\mu = \max\{h_\nu : \int \varphi \, d\nu = \int \varphi \, d\mu\}$

Rotation Sets

Let $\Phi = (\phi_1, ..., \phi_m) : X \to \mathbb{R}^m$ be a continuous potential.

For $\mu \in \mathcal{M}$ the rotation vector of μ is

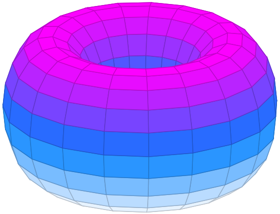
$$\mathrm{rv}(\mu) = \int \Phi \, d\mu = \left(\int \phi_1 \, d\mu, ..., \int \phi_m \, d\mu \right)$$

The rotation set of Φ is

$$\mathrm{Rot}(\Phi) = \{\mathrm{rv}(\mu) : \mu \in \mathcal{M}\}$$

This definition of the rotation set was introduced by Misiurewicz ('89) as a generalization of the Poincaré's rotation number of an orientation preserving homeomorphism on a circle.

Rotation Sets: Intuition



Let $X = \mathbb{T}^m$ be an m-dimensional torus,
$f : \mathbb{T}^m \to \mathbb{T}^m$ be continuous,
$F : \mathbb{R}^m \to \mathbb{R}^m$ be any lifting of f,

$\Phi : \mathbb{T}^m \to \mathbb{R}^m$ be the displacement function,

$$\Phi(x) = F(x) - x$$

If μ is ergodic then

$$\mathrm{rv}(\mu) = \int \Phi \, d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi(f^k(x))$$ for μ-almost all x.

(Birkhoff Ergodic theorem)

$$\frac{1}{n} \sum_{k=0}^{n-1} \Phi(f^k(x)) = \frac{F^n(x) - x}{n}$$ is the average displacement of a point $x \in \mathbb{R}^m$.

$\mathrm{rv}(\mu)$ represents direction and speed of motion of points in X seen by measure μ.

Rotation Sets: Intuition

If $X = \mathbb{T}^2$ and f is a homeomorphism then

all limit points of sequences of the form $\dfrac{F^n(x) - x}{n} = \{\mathrm{rv}(\mu) : \mu \in \mathcal{M}\}$

Misiurewicz, Ziemian ('89)

If X is any metric space, $f : X \to X$ and $\Phi : X \to \mathbb{R}^m$ then

$$\mathrm{conv}\left\{\text{all limit points of sequences of the form } \frac{1}{n}\sum_{k=0}^{n-1}\Phi(f^k(x))\right\} = \{\mathrm{rv}(\mu) : \mu \in \mathcal{M}\}$$

The rotation set describes the asymptotic motion of orbits of the dynamical system.

Rotation Vectors of Ground States

Let X be a compact metric space, $f : X \to X$ and $\Phi : X \to \mathbb{R}^m$ be continuous.
The set of all direction vectors in $\mathbb{R}^m$ is $\{\alpha \in \mathbb{R}^m : \|\alpha\| = 1\} = S^{m-1}$.
Consider a potential $\varphi_\alpha = \alpha \cdot \Phi = \alpha_1\phi_1 + \ldots + \alpha_m\phi_m$

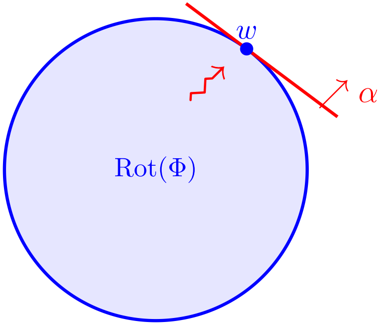
Theorem

Suppose $f : X \to X$ is a continuous map on a compact metric space X such that the entropy map $\mu \mapsto h_\mu$ is upper semi-continuous. For a continuous potential $\Phi : X \to \mathbb{R}^m$ and a direction $\alpha \in S^{m-1}$ let μ_α be a ground state of φ_α. Then

- $\mathrm{rv}(\mu_\alpha) \in \partial \mathrm{Rot}(\Phi)$.
- $\mathrm{rv}(\mu_\alpha) \in H_\alpha$, *where H_α denotes the supporting hyperplane to $\mathrm{Rot}(\Phi)$ with the normal vector α pointing outwards.*
- $h_{\mu_\alpha} = \sup\{h_\nu : \mathrm{rv}(\nu) \in H_\alpha\}$.

Rotation Vectors of Ground States

Question 1: Does any boundary point of $\mathrm{Rot}(\Phi)$ corresponds to a ground state of φ_α for some direction α?



If $\mathrm{Rot}(\Phi)$ is strictly convex, then yes.

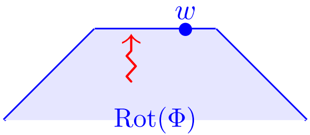
Consider any $w \in \partial\mathrm{Rot}(\Phi)$.

Let H be the unique supporting hyperplane at w. Denote by α its normal vector pointing outwards.

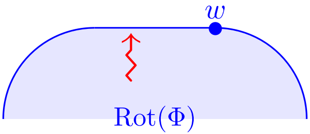
Then all ground states of φ_α have rotation vectors on H, and thus at w.

Rotation Vectors of Ground States

Question 1: Does any boundary point of $\mathrm{Rot}(\Phi)$ corresponds to a ground state of φ_α for some direction α? If $\mathrm{Rot}(\Phi)$ is not strictly convex, then no.



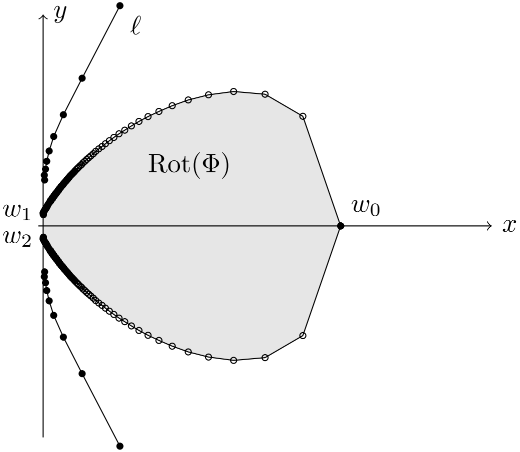
Suppose $w \in \partial\mathrm{Rot}(\Phi)$ is not an extreme point. Then it is not difficult to construct an example where there are no ground states at w.



Suppose $w \in \partial\mathrm{Rot}(\Phi)$ is an extreme point, but the supporting hyperplane at w contains other points of $\mathrm{Rot}(\Phi)$.
It may still happen that there are no ground states at w. Example is much more complicated.

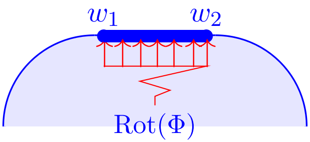
Rotation Vectors of Ground States

Points w_1 and w_2 do not correspond to rotation vectors of any ground state of Φ.



Rotation Vectors of Ground States

Question 2: Can two boundary points of $\mathrm{Rot}(\Phi)$ correspond to ground states of φ_α for the same direction α?



Theorem

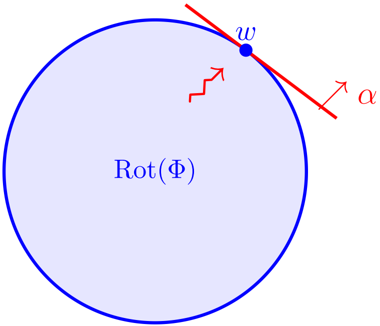
For any direction α the set $\{\mathrm{rv}(\mu) : \mu$ is a ground state of $\varphi_\alpha\}$ is compact and (if $t \mapsto \mu_{t\alpha}$ is continuous) is connected.

Contreras, Lopes, Thieullen ('01) and Jenkinson ('06) established uniqueness of ground states for a generic set of Hölder continuous potentials.

Let $f : X \to X$ be the one-sided full shift over the alphabet $\{0, 1\}$. We construct a Lipschitz continuous potential $\Phi : X \to \mathbb{R}^2$ and a direction vector α such that the set $\{\mathrm{rv}(\mu) : \mu$ is a ground state of $\varphi_\alpha\}$ is a non-trivial compact line segment.

Rotation Vectors of Ground States

Suppose $w \in \partial\mathrm{Rot}(\Phi)$ is an extreme point and the supporting hyperplane at w is unique.



Denote by α the normal vector to this hyperplane pointing outwards.

If μ_α is any ground state of φ_α, then $\mathrm{rv}(\mu_\alpha) = w$.

Question 3: Is the measure μ_α unique?

Answer: No.

We show that for any dimension m there exists a continuous $\Phi : X \to \mathbb{R}^m$ such that w is an extreme point of $\mathrm{Rot}(\Phi)$, the supporting hyperplane at w is unique and we still have multiple ground states at w.

Thank you!