Localized Pressure and Equilibrium States

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Topological Entropy

Let (X, d) be a compact metric space and $f : X \to X$ be a continuous map.

The topological entropy $\mathrm{h}_{top}(f)$ measures the exponential growth rate of distinguishable orbits as we increase the iteration of the map.

To distinguish the orbits we define a new metric on X.

The Bowen metric $\quad d_n(x, y) = \max_{k=0,\dots,n-1} d(f^k(x), f^k(y))$

$F \subset X$ is (n, ϵ)-separated if $d_n(x, y) \geq \epsilon$ for any $x, y \in F$, $x \neq y$

$N(n, \epsilon) = \sup\{\mathrm{card}(F) : F \text{ is } (n, \epsilon)\text{-separated}\}$.

Topological entropy $\quad h_{top}(f) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \epsilon)$

Measure-Theoretic Entropy

Denote by $\mathcal{M}$ the set of all f-invariant probability measures. Let $\mu \in \mathcal{M}$.

The measure-theoretic entropy $h_\mu(f)$ measures the exponential growth rate of statistically significant orbits with respect to μ.

The Variational Principle $h_{top}(f) = \sup\{h_\mu(f) : \mu \in \mathcal{M}\}$

Measures where the supremum is attained are called measures of maximal entropy.

Topological Pressure

Topological entropy $h_{top}(f) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \epsilon)$

$N(n,\epsilon) = \sup\{\mathrm{card}(F) : F \text{ is } (n,\epsilon)\text{-separated}\} = \sup\{\sum_{x \in F} 1 : F \text{ is } (n,\epsilon)\text{-separated}\}$.

$N_\varphi(n,\epsilon) = \sup\{\sum_{x \in F} e^{S_n\varphi(x)} : F \text{ is } (n,\epsilon)\text{-separated}\}$

Here $\varphi : X \to \mathbb{R}$ is a continuous potential and $S_n\varphi(x) = \sum_{k=0}^{n-1} \varphi(f^k(x))$.

Topological pressure $P_{top}(\varphi) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N_\varphi(n, \epsilon)$

The Variational Principle $P_{top}(\varphi) = \sup\{h_\mu + \int_X \varphi d\mu : \mu \in \mathcal{M}\}$

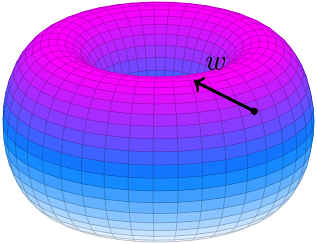
Measures where the supremum is attained are called equilibrium states.

Localized Topological Pressure

Let $\Phi : X \to \mathbb{R}^m$ be a continuous observable. Consider $\frac{1}{n}S_n\Phi(x) = \frac{1}{n}\sum_{k=1}^{n}\Phi(f^k(x))$.

Fix $w \in \mathbb{R}^m$. We are only interested in points $x \in X$ for which $\frac{1}{n}S_n\Phi(x)$ is close to w.

Intuition:



Let $X = \mathbb{T}^m$ be an m-dimensional torus,
$f : \mathbb{T}^m \to \mathbb{T}^m$ be continuous,
$F : \mathbb{R}^m \to \mathbb{R}^m$ be any lifting of f,

$\Phi : \mathbb{T}^m \to \mathbb{R}^m$ be the displacement function,

$$\Phi(x) = F(x) - x$$

Then $\frac{1}{n}\sum_{k=1}^{n}\Phi(f^k(x)) = \frac{F^n(x) - x}{n}$ is the average displacement of a point $x \in \mathbb{R}^m$.

We are interested in points on the torus which on average move in the direction of w.

Localized Topological Pressure

Let $\Phi : X \to \mathbb{R}^m$ be a continuous observable. Consider $\frac{1}{n} S_n \Phi(x) = \frac{1}{n} \sum_{k=1}^{n} \Phi(f^k(x))$.

$$N_\varphi(n, \epsilon, w, r) = \sup\{ \sum_{x \in F} e^{S_n \varphi(x)} : F \text{ is } (n, \epsilon)\text{-separated and } |\tfrac{1}{n} S_n \Phi(x) - w| < r\}$$

Localized topological pressure at w $P_{top}(\varphi, \Phi, w) = \lim_{r \to 0} \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N_\varphi(n, \epsilon, w, r)$

This definition is only meaningful if every neighbourhood of w contains $\frac{1}{n} S_n \Phi(x)$ for arbitrarily large n. The set of such points is called the pointwice rotation set of Φ

$$\mathrm{Rot}_{Pt}(\Phi) = \{w \in \mathbb{R}^m : \forall r > 0 \; \forall N \; \exists n \geq N \; \exists \, x \in X : \; |\tfrac{1}{n} S_n \Phi(x) - w| < r\}$$

If $w \in \mathrm{Rot}_{Pt}(\Phi)$ then there exists $\mu \in \mathcal{M}$ such that $w = \int \Phi d\mu$.

Localized Variational Principle

Denote $\mathcal{M}_\Phi(w) = \{\mu \in \mathcal{M} : w = \int \Phi d\mu\}$ the rotation class of w.

Question: Is it true that $P_{top}(\varphi, \Phi, w) = \sup \left\{h_\mu + \int \varphi d\mu : \mu \in \mathcal{M}_\Phi(w)\right\}$

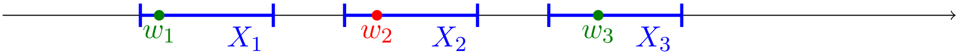
Answer: No (in general) , but it is still true for a wide variety of dynamical systems.

Example 1. $P_{top}(\varphi, \Phi, w) < \sup\left\{h_\mu + \int \varphi d\mu : \mu \in \mathcal{M}_\Phi(w)\right\}$

Take $\varphi \equiv 0$. Idea: If X_1 and X_2 are f-invariant subsets then

$h_{top}(f|_{X_1 \cup X_2}) = \max\{h_{top}(f|_{X_1}), h_{top}(f|_{X_2})\}$,

$h_\mu(f|_{X_1 \cup X_2}) = h_\mu(f|_{X_1})\mu(X_1) + h_\mu(f|_{X_2})\mu(X_2)$



Let $X = X_1 \cup X_2 \cup X_3 \subset \mathbb{R}$. Let $\Phi = \mathrm{id}_X$.Consider a continuous $f : X \to X$ such that X_1, X_2 and X_3 are f-invariant and $h_{top}(f|_{X_1}) = h_{top}(f|_{X_3}) > h_{top}(f|_{X_2})$

Let μ_1 and μ_3 be the entropy maximizing measures for $f|_{X_1}$ and $f|_{X_3}$.
Then $w_1 = \int \Phi d\mu_1 \in X_1$ and $w_3 = \int \Phi d\mu_3 \in X_3$. Pick any $w_2 \in X_2$.
For some $\alpha, \beta \in (0,1)$, $\alpha + \beta = 1$ we have $w_2 = \alpha w_1 + \beta w_3 = \int \Phi d(\alpha\mu_1 + \beta\mu_3)$.

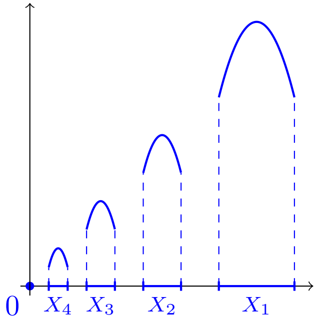
Thus $\sup\left\{h_\mu + \int \varphi d\mu : \mu \in \mathcal{M}_\Phi(w_2)\right\} = h_{\alpha\mu_1 + \beta\mu_3} = h_{top}(f|_{X_1})$.

However, $P_{top}(0, \Phi, w_2) \le h_{top}(f|_{X_2}) < h_{top}(f|_{X_1})$.

Example 2. $P_{top}(\varphi, \Phi, w) > \sup\left\{h_\mu + \int \varphi d\mu : \mu \in \mathcal{M}_\Phi(w)\right\}$

Note that $P_{top}(\varphi, \Phi, w)$ depends on the behavior of the system in small neighbourhoods of w, but the supremum is taken over measures whose integrals are exactly w.

Take $\varphi \equiv 0$. Consider a sequence of disjoint compact intervals $X_n \subset \mathbb{R}$, $X_n \to \{0\}$.



Let $X = \cup_{n=1}^{\infty} X_n \cup \{0\}$ and $\Phi = \mathrm{id}_X$.
Define $f|_{X_n}$ to be topologically conjugate to the logistic map $g(x) = 4x(1-x)$ on $[0, 1]$ and $f(0) = 0$.

Then $h_{top}(f|_{X_n}) = \log 2$. Let μ_n be the ergodic entropy maximising measure for $f|_{X_n}$.
Since in any neighbourhood of $w = 0$ there is μ_n with $h_{\mu_n} = \log 2$, we have $P_{top}(0, \mathrm{id}_X, 0) = \log 2$.

However, $w = 0$ is a fixed point of f and extreme point of X
The only measure in $\mathcal{M}_\Phi(0)$ is the point-mass measure at 0.
We obtain $\sup\left\{h_\mu + \int \varphi d\mu : \mu \in \mathcal{M}_\Phi(0)\right\} = 0$

Localized Variational Principle

To prove $P_{top}(\varphi, \Phi, w) = \sup\left\{h_\mu + \int \varphi d\mu : \mu \in \mathcal{M}_\Phi(w)\right\}$ we need additional assumptions.

Example 2: We have $w_n = \int \Phi d\mu_n \to 0, \quad \sup\left\{h_\mu + \int \varphi d\mu : \mu \in \mathcal{M}_\Phi(w_n)\right\} = \log 2$, but $\sup\left\{h_\mu + \int \varphi d\mu : \mu \in \mathcal{M}_\Phi(0)\right\} = 0$

Assumption 1: The function $v \mapsto \sup\left\{h_\mu + \int \varphi d\mu : \mu \in \mathcal{M}_\Phi(v)\right\}$ is continuous at w.

This condition holds if the entropy map $\mu \mapsto h_\mu$ is upper semi-continuous.

Example 1: $\sup\left\{h_\mu + \int \varphi d\mu : \mu \in \mathcal{M}_\Phi(w_2)\right\}$ is not attained at an ergodic measure. If μ is ergodic and $\int d\mu = w_2$ then $\mu(X_1) = \mu(X_3) = 0$ and $h_\mu \le h_{top}(f|_{X_2}) < h_{\alpha\mu_1 + \beta\mu_2}$.

Assumption 2: $\sup\left\{h_\mu + \int \varphi d\mu : \mu \in \mathcal{M}_\Phi(w)\right\}$ is approximated by ergodic measures. Precisely, there is (μ_n)-ergodic, such that

$$\int \Phi d\mu_n \to w \quad \text{and} \quad h_{\mu_n} + \int \varphi d\mu_n \to \sup\left\{h_\mu + \int \varphi d\mu : \mu \in \mathcal{M}_\Phi(w)\right\}$$

Localized Variational Principle

Theorem

Let $f : X \to X$ be a continuous map on a compact metric space X.
Let $\Phi : X \to \mathbb{R}^m$ and $\varphi : X \to \mathbb{R}$ be continuous and let $w \in \mathrm{Rot}_{Pt}(\Phi)$ such that

1. $v \mapsto \sup\left\{h_\mu(f) + \int_X \varphi\, d\mu : \mu \in \mathcal{M}_\Phi(v)\right\}$ *is continuous at w;*
2. $\sup\left\{h_\mu(f) + \int_X \varphi\, d\mu : \mu \in \mathcal{M}_\Phi(w)\right\}$ *is approximated by ergodic measures.*

Then

$$P_{top}(\varphi, \Phi, w) = \sup\left\{h_\mu(f) + \int_X \varphi\, d\mu : \mu \in \mathcal{M} \text{ and } \int_X \Phi\, d\mu = w\right\}$$

This theorem holds for a wide variety of systems and potentials such as expansive homeomorphisms with specification, which include topological mixing two-sided subshifts of finite type as well as diffeomorphisms with a locally maximal topological mixing hyperbolic set.

Localized Equilibrium States

Assume that for (X, f) the localized variational principle holds, e.i.

$$P_{top}(\varphi, \Phi, w) = \sup \left\{ h_\mu(f) + \int_X \varphi \, d\mu : \mu \in \mathcal{M}_\Phi(w) \right\}$$

Fix $w \in \mathrm{Rot}_{Pt}(\Phi)$. We say that $\mu \in \mathcal{M}_\Phi(w)$ is a localized equilibrium state of $\varphi \in C(X, \mathbb{R})$ with respect to Φ and w if

$$P_{top}(\varphi, \Phi, w) = h_\mu + \int \varphi d\mu$$

If the entropy map $\mu \mapsto h_\mu$ is upper semi-continuous, there exists at least one localized equilibrium state.

Classical case: If the system is "nice", the equilibrium state is unique.

Localized Equilibrium States

Classical Equilibrium States: Let (X, f) be expansive homeomorphisms with specification (include topological mixing subshifts of finite type).
For any Holder $\varphi : X \to \mathbb{R}$ there is a unique equilibrium state μ_φ (ergodic) such that

$$P_{top}(\varphi) = h_{\mu_\varphi} + \int \varphi d\mu_\varphi$$

Localized Equilibrium States: Localized equilibrium state at a point is not unique (even for shift spaces).
We construct a Lipschitz continuous Φ on a shift map exhibiting infinitely many ergodic localized equilibrium states for $\varphi \equiv 0$.

Thank you