Multiple phase transitions on compact symbolic systems

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Topological Pressure and Equilibrium States

Let $\phi : X \to \mathbb{R}$ be a continuous potential associated with a symbolic dynamical system (X, T) over a finite alphabet.

The topological pressure of ϕ is defined by

$$P_{\text{top}}(\phi) = \sup_{\mu \in \mathcal{M}} \{\mathsf{h}_\mu + \int \phi \, d\mu\},$$

where $\mathcal{M}$ is the set of all T-invariant probability measures and h_μ is the measure-theoretic entropy of μ.

From the statistical physics point of view, $P_{\text{top}}(\phi)$ corresponds to the minimum of the free energy $E_\mu = -(\mathsf{h}_\mu + \int \phi \, d\mu)$.

A measure $\mu \in \mathcal{M}$ which minimizes the free energy (i.e. $P_{\text{top}}(\phi) = \mathsf{h}_\mu + \int \phi \, d\mu$) is called an equilibrium state for ϕ.

If the map $\mu \mapsto h_\mu$ is upper semi-continuous, then there exists at least one equilibrium state. (True for subshifts of finite type).

Phase Transitions

We introduce a parameter $\beta > 0$ (interpreted as the inverse temperature of the system) and study the equilibrium states of the potential $\beta\phi$.

When the temperature changes, the equilibrium of the system changes as well.
A phase transition refers to a qualitative change of the properties of a dynamical system as a result of the change in temperature.
Intuitively, this means co-existance of several equilibria at the same temperature.

We are interested in the values of β for which potential $\beta\phi$ has more than one equilibrium state.

Connection to Pressure Function

Co-existence of several equilibria vs. regularity of the pressure:

- P_{top} is Gateaux differentiable at $\phi \iff \phi$ has a unique equilibrium state
- If the pressure function $\beta \mapsto P_{\text{top}}(\beta\phi)$ is not differentiable at β_0 then $\beta_0\phi$ has at least two equilibrium states.
- Non-uniqueness of equilibrium states for $\beta_0\phi$ does not imply non-differentiability of $P_{\text{top}}(\beta\phi)$ at β_0.
  Leplaideur (2015): there is a continuous ϕ on a mixing subshift of finite type such that $P_{\text{top}}(\beta\phi)$ is analytic on some interval, but uniqueness of equilibrium states fails for two distinct values of β in that interval.
- $P_{\text{top}}(\beta\phi)$ in not differentiable at $\beta_0 \iff \beta_0\phi$ has two equilibrium states with distinct entropies.

Lack of Phase Transitions

We say ϕ has a phase transition at β_0 if the pressure function $\beta \mapsto P_{\rm top}(\beta\phi)$ is not differentiable at β_0 (first order phase transition).

Ruelle (1968): If X is a transitive subshift of finite type then the pressure functional $P_{\rm top}$ acts real analytically on the space of Hölder continuous potentials.

In particular, when ϕ is Hölder

- the pressure function $\beta \mapsto P_{\rm top}(\beta\phi)$ is analytic,
- $\beta\phi$ has a unique equilibrium state for any β,

and hence there are no phase transitions.

In order to allow the possibility of phase transitions one needs to consider potential functions that are merely continuous.

To the best of our knowledge there are no examples in the literature with more than two phase transitions.

The Main Result

We develop a method to explicitly construct a continuous potential with any (finite or infinite) number of first order phase transitions occurring at any sequence of predetermined points.

Theorem

Let X be a two-sided full shift on two symbols. Then for any given increasing sequence of positive real numbers $\{\beta_n\}$ there is a continuous potential $\phi : X \to \mathbb{R}$ which has phase transitions precisely at β_n.

Since the pressure function $\beta \mapsto P_{\text{top}}(\beta\phi)$ is Lipschitz and convex, at most countably many phase transitions are possible. Taking $\{\beta_n\}$ to be infinite we see that the case of infinitely many phase transitions can indeed be realized.

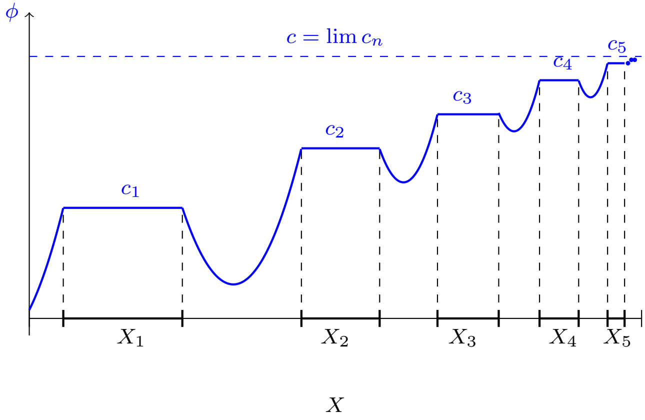
When $\{\beta_n\}$ is finite, we have a "freezing" phase transition at $\beta = \beta_N$. Physically, this means that for some positive temperature $1/\beta_N$, the systems reaches its unique ground state and then ceases to change.

General Idea

To construct $\phi : X \to \mathbb{R}$

- Fix a positive strictly increasing sequence (β_n).
- Take a sequence (X_n) of disjoint subshifts of finite type in X.
- For a suitable sequence of values (c_n) set ϕ to be constant c_n on each X_n and $c = \lim c_n$ on accumulation points of $\bigcup X_n$.
  <u>We need:</u> $P_{\text{top}}(\beta_n \phi|_{X_n}) = P_{\text{top}}(\beta_n \phi|_{X_{n+1}})$ and $P_{\text{top}}(\beta_n \phi|_{X_k}) < P_{\text{top}}(\beta_n \phi|_{X_n})$ whenever $k \notin \{n, n+1\}$.
- Make ϕ drop sharply outside $\bigcup X_n$ and force the equilibrium measures at all values of β to be supported on $\bigcup X_n$.

Visual Aid



Issues:

- Continuity of ϕ
- Estimates on the pressure (!)

The main difficulty is to ensure that the drop-off is sufficiently steep so that for any ergodic μ not supported on $\bigcup X_k$ we have $h_\mu + \beta_n \int \phi \, d\mu < P_{\mathrm{top}}(\beta_n \phi|_{X_n})$.

Our Technique

Our Technique:

- For each $x \in X$ we look for blocks within x from X_ns
- We note their locations and sizes.
- To store this data we introduce an additional subshift $Z \subset \{0,1\}^{\mathbb{Z}}$ and consider $X \times Z$.
  We call Z the pin-sequence space since for a pair (x, z) a 1 in z pins exactly the place in x where one block from $\bigcup X_n$ ends and another one begins.
- We define $\phi(x)$ based on the information from Z.
- All the estimates on the pressure are performed on $X \times Z$ and then projected back to X.

Thank you!