Realization of Anosov Diffeomorphisms on the Torus

Tamara Kucherenko, CCNY (joint work with Anthony Quas)

April 5, 2025

This work was largely developed during our one-weak stay at the *Centre International de Rencontres Mathématiques* in Luminy, France through the *Research in Residence* program. We thank CIRM for the support and hospitality.

Anosov Diffeomorphisms

Consider the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be an Anosov diffeomorphism, i.e.

- there is a continuous splitting of the tangent bundle of $\mathbb{T}^2$ into a direct sum $E^u \oplus E^s$ which is preserved by the derivative Df;
- the unstable subbundle E^u is uniformly expanded by Df and the stable subbundle E^s is uniformly contracted by Df;

Example: Let A be a 2×2 matrix with integer entries, determinant ± 1, and no eigenvalues of absolute value one. The map $L_A : \mathbb{T}^2 \to \mathbb{T}^2$ induced by A is called a hyperbolic toral automorphism

Any Anosov diffeomorphism f is homotopic and topologically conjugate to a hyperbolic automorphism L.

Franks (1969): if the non-wandering set is $\mathbb{T}^2$ $(\mathbb{T}^n)$
Newhouse (1970): if either $\dim E^s = 1$ or $\dim E^u = 1$
Manning (1974): any Anosov on $\mathbb{T}^n$

Regularity of the Conjugacy

Fix a hyperbolic automorphism L on $\mathbb{T}^2$.

Let f and g be $C^r(r > 1)$ Anosov diffeomorphisms in the homotopy class of L. Then there is a homeomorphism h such that $h \circ f = g \circ h$.

Smooth Conjugacy Problem: When h has the same regularity as f and g?

Anosov (1967): *h can be merely Hölder even for high regularity maps.*

Llave, Marco, Moriyón (1980s): *If f and g are C^∞ and the Lyapunov exponents at corresponding periodic orbits are the same, then h is C^∞* (later similar results were obtained for C^r)

If h were C^1, then when x is of period p for f, $h(x)$ is of period p for g and $Df^p(x) = Dh^{-1}(h(x))Dg^p(h(x))Dh(x)$, so the Lyapunov exponents match!

F. Rodriguez Hertz: *Can the equality of the Lyapunov exponents be replaced by the equality of the pressure functions of the geometric potentials?*

Pressure Function

Let $\phi : \mathbb{T}^2 \to \mathbb{R}$ be a continuous potential associated with a dynamical system $(\mathbb{T}^2, f)$.

The topological pressure of ϕ with respect to f is $P_{\text{top}}(f, \phi) = \sup_\mu \{\mathsf{h}_\mu(f) + \int \phi \, d\mu\}$, where μ runs over the set of all f-invariant probability measures on $\mathbb{T}^2$ and h_μ is the measure-theoretic entropy of μ.

A measure μ which realizes the supremum is called an equilibrium state of ϕ.

Bowen (1975): when f is Anosov and ϕ is Hölder, the equilibrium state is unique.

Equilibrium states are mathematical generalizations of Gibbs distributions in statistical physics

Important ones are: the measure of maximal entropy (ϕ is a constant potential),
the SRB measure (ϕ is the geometric potential).

The geometric potential is the negative logarithm of the unstable Jacobian of f,

$$\phi_f^u(x) = -\log \left| D_u f(x) \right|.$$

The pressure function of ϕ is the map $t \mapsto P(f, t\phi)$, where t is a real parameter.

Dynamical information, e.g. Lyapunov exponents, is encoded into the pressure function of ϕ_f^u.

Question

Question [Erchenko 2020, attr. to F. Rodriguez Hertz]

Let f and g be C^∞ area-preserving Anosov diffeomorphisms on $\mathbb{T}^2$ that are homotopic. Assume $P_{\text{top}}(f, t\phi_f^u) = P_{\text{top}}(g, t\phi_g^u)$ for all t. Does this imply that f and g are C^∞ conjugate?

Yes, when one diffeomorphism is an automorphism!

- f and g are C^∞ conjugate $\Leftrightarrow$ Lyapunov exponents at corresponding periodic orbits are the same
- Lyapunov exponents $\sim$ $-\log \left| D_u f^m \right| = S_m\, \phi_f^u$ (Birkhoff sums of ϕ_f^u)

Does $P_{\text{top}}(f, t\phi_f^u) = P_{\text{top}}(g, t\phi_g^u)$ imply that the Birkhoff sums at corresponding periodic orbits are the same?

Symbolic Coding

Bowen (1975): Every Anosov f can be represented as a subshift of finite type Ω.

Precisely, using Markov partition of $\mathbb{T}^2$ one can find a finite set $\mathcal{A}$ (the set of rectangles of the Markov partition) and a mixing subshift of finite type $\Omega \subset \mathcal{A}^{\mathbb{Z}}$ such that there exists a finite-to-one factor map $\pi : \Omega \to \mathbb{T}^2$ which is Hölder.

Then $\phi_f^u \circ \pi$ is a Hölder potential on Ω.

What happens to Hölder potentials on shifts?

Hölder Potentials on Shifts

Suppose (Ω, σ) is a subshift of finite type and $\psi : \Omega \to \mathbb{R}$ is a Hölder potential.

The Birkhoff sum of ψ is $S_m\psi(x) = \sum_{k=0}^{m-1} \psi(\sigma^k x)$

The set $\{(S_m\psi(x), m) : \sigma^m x = x\}$ is called the unmarked orbit spectrum of ψ.

Topological definition of the pressure function:

$$P_{\text{top}}(t\psi) = \lim_{m\to\infty} \frac{1}{m} \log \left(\sum_{\sigma^m x = x} e^{tS_m\psi(x)} \right)$$

For subshifts of finite type this definition is equivalent to the measure-theoretic one from before.

ψ and ψ' have the same unmarked orbit spectrum $\Rightarrow$ $P_{\text{top}}(t\psi) = P_{\text{top}}(t\psi')$
$\Leftarrow (?)$

Hoölder Functions on Shifts

Pollicott, Weiss (2003): *There exists an uncountable family of Hölder continuous functions on a mixing subshift of finite type with different unmarked orbit spectra, but all sharing the same pressure function.*

Are Hölder functions arising from geometric potentials of Anosov diffeomorphisms on the torus special enough that the equality of their pressure functions implies the equality of their unmarked orbit spectrum?

No, they are not that special

Realization of Anosov Diffeomorphisms on the Torus

We show that the set of pressure functions for Anosov diffeomorphisms with respect to the geometric potential is equal to the set of pressure functions for the hyperbolic automorphism with respect to Hölder potentials.

Theorem 1

Let L be a hyperbolic automorphism of $\mathbb{T}^2$ and let μ be the equilibrium state for a Hölder continuous potential ϕ with $P_{\text{top}}(L, \phi) = 0$. Then there exists a $C^{1+\alpha}$ area-preserving Anosov diffeomorphism f of $\mathbb{T}^2$ such that

- *the system $f\colon (\mathbb{T}^2, \text{Leb}) \to (\mathbb{T}^2, \text{Leb})$ is conjugate to $L\colon (\mathbb{T}^2, \mu) \to (\mathbb{T}^2, \mu)$ by a map h*
- *$-\log |D_u f| \circ h$ is cohomologous to ϕ.*

Related work by Cawley (1993) establishes a bijection between Teichmüller space of an Anosov diffeomorphism and the quotient of Hölder functions by the subspace of almost coboundaries.

Idea of the Proof

Our proof is explicit and constructive:

Start with a hyperbolic automorphism $L : \mathbb{T}^2 \to \mathbb{T}^2$, a Hölder potential $\phi : \mathbb{T}^2 \to \mathbb{R}$ with $P_{\text{top}}(L, \phi) = 0$, and an equilibrium state μ of ϕ.

Setep 1: Build a new $C^{1+\alpha}$ atlas on $\mathbb{T}^2$ and show that with respect to this new atlas L is differentiable with Hölder derivative.

Setep 2: Verify that the equilibrium state is a volume measure with respect to the new atlas.

Setep 3: Show that ϕ is cohomologous to the geometric potential where the unstable derivative is computed using the new atlas.

Family of Charts

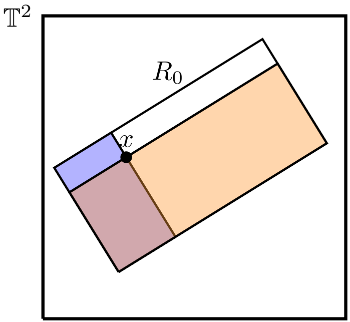
L is the hyperbolic automorphism,
μ is the equilibrium state of the Hölder potential ϕ.

Let R_0 be the first rectangle in the Markov partition. Pick $x \in R_0$

Draw the stable manifold of x within R_0.
Define $\xi_1(x)$ to be the μ-measure of the blue region.

Draw the unstable manifold of x within R_0.
Define $\xi_2(x)$ to be the μ-measure of the orange region. Let $\xi(x) = (\xi_1(x), \xi_2(x))$.

The family of charts $(\mathrm{Int}(R_0) - v, \xi \circ \tau_v)$ where v is homoclinic and τ_v is the translation by v.



Conjugacy Question

Does $P_{\text{top}}(f, t\phi_f^u) = P_{\text{top}}(g, t\phi_g^u)$ imply that the Birkhoff sums at corresponding periodic orbits are the same?

In view of Theorem 1, we need to find Hölder potentials having identical pressure functions with respect to an automorphism L, but different unmarked orbit spectra.

Let Ω be a shift representing L and $\pi\colon \Omega \to \mathbb{T}^2$ be the factor map.
One can have uncountably many Hölder potentials on Ω with this property.

If ϕ is Hölder on the torus then $\phi \circ \pi$ is Hölder on the shift. Converse is not true!
We have to look for torus-continuous examples.

Theorem 2

There exist homotopic $C^{1+\alpha}$ area-preserving Anosov diffeomorphisms f and g on $\mathbb{T}^2$ such that $P_{\text{top}}(f, t\phi_f^u) = P_{\text{top}}(g, t\phi_g^u)$ for all t, but f and g fail to be C^1 conjugate.

Related Questions

A.Erchenko, *Flexibility of Lyapunov exponents*, ETDS (2022):

Question 5. *Let L be an Anosov automorphism on $\mathbb{T}^2$. Given a strictly convex real analytic function $F\colon \mathbb{R} \to \mathbb{R}$ such that $F(0) = h_{top}(L)$, $F(1) = 0$, $\frac{dF}{dt}|_{t=0} < -h_{top}(L)$, $\frac{dF}{dt}|_{t=1} \in (-h_{top}(L), 0)$, and $F(t)$ has asymptotes as $t \to \pm\infty$. Does there exist a smooth area-preserving Anosov diffeomorphism f homotopic to L such that $P(\phi_t^f) = F(t)$?*

The answer to the above question will require different techniques than presented in this paper. If the answer is negative then it would be interesting to determine which extra conditions on the function must be satisfied. For example, do the higher derivatives of the pressure function [KS01] provide any additional restrictions? Is there a finite list of conditions that must be added in order to obtain flexibility?

We can also consider a rigidity problem connected to the pressure function. Let f be a smooth area-preserving Anosov diffeomorphism homotopic to an Anosov automorphism L. By work of de la Llave [dlL87] and of Marco and Moriyón [MM87], we have that if $\lambda_{abs}(f) = h_{top}(L)$, then f and L are C^∞ conjugate. By the properties of the pressure functions discussed above, we have that if $P(\phi_t^f) = P(\phi_t^L)$ for all $t \in \mathbb{R}$, then f and L are C^∞ conjugate. A natural question is if L can be replaced by any smooth area-preserving Anosov diffeomorphism.

Question 6. *Let f and g be smooth area-preserving Anosov diffeomorphisms on $\mathbb{T}^2$ that are homotopic. Assume $P(\phi_t^f) = P(\phi_t^g)$ for all $t \in \mathbb{R}$. Does this imply that f and g are C^∞ conjugate?*

Flexibility Question

Question [Erchenko 2020, the same page as Federico's question!]

Suppose $L : \mathbb{T}^2 \to \mathbb{T}^2$ is a hyperbolic automorphism and $F : \mathbb{R} \to \mathbb{R}$ is a strictly convex real analytic asymptotically linear function satisfying
$F(0) = h_{\text{top}}(L)$, $F(1) = 0$, and $\frac{dF}{dt}|_{t=0} < -h_{\text{top}}(L) < \frac{dF}{dt}|_{t=1} < 0$.
Does there exists a smooth Anosov area-preserving diffeomorphism f homotopic to L for which $P_{\text{top}}(f, t\phi_f^u) = F(t)$?

Rationale:

- $-\frac{dP}{dt}|_{t=0} = \lambda_{\text{mme}}(f)$, $-\frac{dP}{dt}|_{t=1} = \lambda_{\text{Leb}}(f)$, $\lambda_{\text{Leb}}(f) \leq h(f) = h(L) \leq \lambda_{\text{mme}}(f)$
- Erchenko proved that one can pick any λ_{Leb} and λ_{mme}, so we can vary the slopes.
- The only properties of the pressure function known at the time were analiticity, strict convexity and the existence of asymptotes.

Answer: No.

Pressure Functions on Shifts

Suppose Ω is a mixing subshift of finite type and $\phi : \Omega \to \mathbb{R}$ is a Hölder potential.

K., Quas (2023):

- There is an exponential lower bound on the gap between $P_{\text{top}}(t\phi)$ and its asymptote.
- There is a quantitative lower bound on the curvature of $P_{\text{top}}(t\phi)$.

Ma, Pollicott, (2023): There is a relation between the higher derivatives of $P_{\text{top}}(t\phi)$

Are there any other properties of the pressure functions for Hölder potentials on shifts?

What about Hölder potentials on the torus?

Questions

Question 1

Does the set of the pressure functions of Hölder potentials on the two-dimensional torus coinside with the set of pressure functions for Hölder potentials on mixing subshifts of finite type?

Question 2

Is it possible to give a nice characterization of the functions which arise as pressure functions for Hölder potentials on mixing subshifts of finite type?

Happy Birthday, Yunping!